Appendix

EC.1. Closed-Form Solutions for Symmetric Equilibria

In this Appendix, we derive closed-form expressions for the symmetric follower reactions, symmetric leader reactions, the symmetric forward market equilibria, and the symmetric Stackelberg equilibria. These results are used to derive the structural results in Section 4. We denote by α_x and α_y the normalized leader and follower demands respectively, and by $\triangle C$ the normalized marginal cost gap:

$$\alpha_x = \frac{1}{\beta}(\alpha - C),$$
$$\alpha_y = \frac{1}{\beta}(\alpha - c),$$
$$\triangle C = \frac{1}{\beta}(c - C).$$

We use the following notation. For scalars $z, a, b \in \mathbb{R}$ such that $a \leq b$, let

$$[z]_a^b := \begin{cases} a, & \text{if } z \leq a, \\ b, & \text{if } z \geq b, \\ z, & \text{otherwise.} \end{cases}$$

We will use the following properties:

(i) For any $c \in \mathbb{R}$, $c + [z]_a^b = [z+c]_{a+c}^{b+c}$.

(ii) If $c > 0$, then $c\,[z]_a^b = [cz]_{ca}^{cb}$.

(iii) If $c < 0$, then $c\,[z]_a^b = [cz]_{cb}^{ca}$.

EC.1.1. Spot Market Analyses

Proposition EC.1. *Fix a follower $l \in N$ and suppose $f_j = f$ for every $j \neq l$. There is a unique Nash equilibrium $\mathbf{y}$ in the spot market such that, for each $j \neq l$,*

$$y_j = \left[\frac{1}{N} \left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l \right) \right]_0^k. \tag{EC.1}$$

Proof. The uniqueness of the Nash equilibrium follows from Theorem 5 of Jing-Yuan and Smeers (1999). Each follower $j \in N$ has a strategy set $[0,k]$ which is compact. Its payoff function in the spot market $\phi_j^{(s)}$ is continuous in all arguments and is strictly concave in y_j. Thus, from the Karush-Kuhn-Tucker (KKT) conditions, we infer that $\mathbf{y} \in [0,k]^N$ is a Nash equilibrium of the spot market, if and only if there exists $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}_+^N$ such that, for each $j \in N$:

$$\nabla_{y_j}\left[\phi_j^{(s)}(y_j;\mathbf{y}_{-j}) + \lambda_j y_j + \mu_j(k-y_j)\right] = 0, \tag{EC.2}$$

$$\lambda_j y_j = \mu_j(k-y_j) = 0. \tag{EC.3}$$

Take any $j \neq l$. Expanding the LHS of (EC.2) gives:

$$\begin{aligned}\nabla_{y_j}\left[\phi_j^{(s)}(y_j;\mathbf{y}_{-j}) + \lambda_j y_j + \mu_j(k-y_j)\right] &= \beta\left(\alpha_y + f - \sum_{i=1}^M x_i - y_j - \sum_{j'=1}^N y_{j'}\right) + \lambda_j - \mu_j \\ &= \beta\left(\alpha_y + f - \sum_{i=1}^M x_i - y_l - N y_j\right) + \lambda_j - \mu_j.\end{aligned}$$

Suppose $0 < y_j < k$. Then (EC.3) imply that $\lambda_j = \mu_j = 0$. From (EC.2), we obtain

$$y_j = \frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^M x_i - y_l\right). \tag{EC.4}$$

Suppose $y_j = 0$. Then (EC.3) imply that $\mu_j = 0$. From (EC.2), we obtain

$$-\left(\alpha_y + f - \sum_{i=1}^M x_i - y_l\right) = \lambda_j \geq 0. \tag{EC.5}$$

Suppose $y_j = k$. Then (EC.3) imply that $\lambda_j = 0$. From (EC.2), we obtain

$$\left(\alpha_y + f - \sum_{i=1}^M x_i - y_l - Nk\right) = \mu_j \geq 0. \tag{EC.6}$$

Since $0 \leq y_j \leq k$, (EC.4) – (EC.6) together imply that

$$y_j = \begin{cases} 0, & \text{if } \frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^M x_i - y_l\right) \leq 0, \\ k, & \text{if } \frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^M x_i - y_l\right) \geq k, \\ \frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^M x_i - y_l\right), & \text{otherwise,} \end{cases}$$

which is equivalent to (EC.1).

Proposition EC.2. *Suppose $f_j = f$ for every $j \in N$. There is a unique Nash equilibrium in the spot market, given by*

$$y_j = \left[\frac{1}{N+1}\left(\alpha_y + f - \sum_{i=1}^{M} x_i\right)\right]_0^k. \tag{EC.7}$$

Proof. The uniqueness of the Nash equilibrium follows from Theorem 5 of Jing-Yuan and Smeers (1999). Thus, it suffices to show that the given productions form a Nash equilibrium. From the optimality conditions in (EC.2) – (EC.3), we infer that $y \in [0,k]$ is a symmetric Nash equilibrium in the spot market, if and only if there exists scalars $\lambda, \mu \in \mathbb{R}_+$ such that,

$$\beta\left(\alpha_y + f - \sum_{i=1}^{M} x_i - (N+1)y\right) + \lambda - \mu = 0,$$

$$\lambda y = \mu\,(k-y) = 0.$$

Let

$$\lambda = \left[-\beta\left(\alpha_y + f - \sum_{i=1}^{M} x_i - (N+1)y\right)\right]_0^\infty,$$

$$\mu = \left[\beta\left(\alpha_y + f - \sum_{i=1}^{M} x_i - (N+1)y\right)\right]_0^\infty.$$

It is straightforward to show that y defined in (EC.7), and λ, μ defined above, together satisfy the optimality conditions.

EC.1.2. Follower Reaction Analyses

Proposition EC.3. *Fix the leaders' productions $\mathbf{x} \in \mathbb{R}_+^M$. Let $F \subseteq \mathbb{R}$ denote the set of symmetric follower reactions, i.e., for each $f \in F$ and $j \in N$,*

$$\phi_j\left(f; f\mathbf{1}, \mathbf{x}\right) \geq \phi_j\left(\bar{f}; f\mathbf{1}, \mathbf{x}\right), \quad \forall \bar{f} \in \mathbb{R}. \tag{EC.8}$$

Let $\xi := \alpha_y - \sum_{i=1}^M x_i$. Then,

$$F = \begin{cases} (-\infty, -\xi]\,, & \text{if } \xi < 0, \\ \left\{\frac{N-1}{N^2+1}\xi\right\}, & \text{if } 0 \leq \xi \leq \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k, \\ \varnothing, & \text{if } \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k < \xi < (N+1)\,k, \\ [-\xi + (N+1)k, \infty)\,, & \text{if } (N+1)\,k \leq \xi. \end{cases} \tag{EC.9}$$

Moreover, for each $f \in F$,

$$\begin{aligned} y_j(f\mathbf{1},\mathbf{x}) = 0 &\iff \xi \le 0, \\ 0 < y_j(f\mathbf{1},\mathbf{x}) < k &\iff 0 < \xi \le \frac{(N^2+1)(N-1)}{N^2 - 2\sqrt{N}+1}k, \\ y_j(f\mathbf{1},\mathbf{x}) = k &\iff (N+1)k \le \xi. \end{aligned}$$

Proof. The proof proceeds in three steps. In step 1, we reformulate a follower's payoff maximization problem into a problem involving its production quantity only. In step 2, we compute its payoff maximizing production quantity. In step 3, we compute the symmetric follower forward positions that satisfy the condition that every follower is producing at its payoff maximizing quantity. The latter gives the set of symmetric follower reactions.

Step 1: Fix a follower $l \in N$ and suppose $f_j = f$ for every $j \neq l$. Using Proposition EC.1 to substitute for $y_j(f_j;\mathbf{f}_{-j},\mathbf{x})$ for every $j \neq l$, we infer that the total production in the spot market is given by

$$\begin{aligned} \sum_{j=1}^{N} y_j(f_j;\mathbf{f}_{-j},\mathbf{x}) &= y_l(f_l;f\mathbf{1},\mathbf{x}) + (N-1)\left[\frac{1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l(f_l;f\mathbf{1},\mathbf{x})\right)\right]_0^k \\ &= y_l(f_l;f\mathbf{1},\mathbf{x}) + \left[\frac{N-1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i - y_l(f_l;f\mathbf{1},\mathbf{x})\right)\right]_0^{(N-1)k} \\ &= \left[\frac{N-1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i\right) + \frac{1}{N} y_l(f_l;f\mathbf{1},\mathbf{x})\right]_{y_l(f_l;f\mathbf{1},\mathbf{x})}^{y_l(f_l;f\mathbf{1},\mathbf{x})+(N-1)k}, \end{aligned}$$

By substituting the above into follower l's payoff, and using the fact that $y_l(\mathbb{R};f\mathbf{1},\mathbf{x}) = [0,k]$, we obtain

$$\sup_{f_l\in\mathbb{R}} \phi_l(f_l;f\mathbf{1},\mathbf{x}) = \sup_{f_l\in\mathbb{R}} \left(P\left(\left[\frac{N-1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i\right) + \frac{1}{N} y_l(f_l;f\mathbf{1},\mathbf{x})\right]_{y_l(f_l;f\mathbf{1},\mathbf{x})}^{y_l(f_l;f\mathbf{1},\mathbf{x})+(N-1)k} + \sum_{i=1}^{M} x_i\right) - c\right)\cdot y_l(f_l;f\mathbf{1},\mathbf{x}) \tag{EC.10}$$

$$= \sup_{y\in[0,k]} \hat{\phi}_l(y;f,\mathbf{x}), \tag{EC.11}$$

where

$$\hat{\phi}_l(y;f,\mathbf{x}) := \left(P\left(\left[\frac{N-1}{N}\left(\alpha_y + f - \sum_{i=1}^{M} x_i\right) + \frac{1}{N} y\right]_{y}^{y+(N-1)k} + \sum_{i=1}^{M} x_i\right) - c\right)\cdot y.$$

Step 2: We solve for the solution to (EC.11). Substituting for the demand function yields

$$\hat{\phi}_l(y;f,\mathbf{x}) = \beta\left(\xi - \left[\frac{N-1}{N}(\xi+f)+\frac{1}{N}y\right]_y^{y+(N-1)k}\right)y$$

$$= \begin{cases} \beta(\xi-y)\,y, & \text{if (EC.12a) holds,}\\ \begin{cases}\beta\left(\frac{1}{N}\xi-\frac{N-1}{N}f-\frac{1}{N}y\right)y, & \text{if } 0\le y<\xi+f,\\ \beta(\xi-y)\,y, & \text{if } k\ge y\ge \xi+f,\end{cases} & \text{if (EC.12b) holds,}\\ \beta\left(\frac{1}{N}\xi-\frac{N-1}{N}f-\frac{1}{N}y\right)y, & \text{if (EC.12c) holds,}\\ \begin{cases}\beta(\xi-y-(N-1)k)\,y, & \text{if } 0\le y\le \xi+f-Nk,\\ \beta\left(\frac{1}{N}\xi-\frac{N-1}{N}f-\frac{1}{N}y\right)y, & \text{if } k\ge y>\xi+f-Nk,\end{cases} & \text{if (EC.12d) holds,}\\ \beta(\xi-y-(N-1)k)\,y, & \text{if (EC.12e) holds,}\end{cases}$$

where the second equality follows from the fact that $y\in[0,k]$ and the five cases (EC.12a) – (EC.12e) are defined by

$$\xi+f\le 0, \tag{EC.12a}$$

$$0<\xi+f<k, \tag{EC.12b}$$

$$k\le\xi+f\le Nk, \tag{EC.12c}$$

$$Nk<\xi+f<(N+1)k, \tag{EC.12d}$$

$$(N+1)k\le\xi+f. \tag{EC.12e}$$

We analyze each case separately.

Case (i): $\xi+f\le 0$. Then $\hat{\phi}_l(y;f,\mathbf{x})$ is a smooth function in y over the interval $[0,k]$. The first and second derivatives are given by

$$\frac{\partial}{\partial y}\hat{\phi}_l(y;f,\mathbf{x}) = \beta(\xi-2y),$$
$$\frac{\partial^2}{\partial y^2}\hat{\phi}_l(y;f,\mathbf{x}) = -2\beta<0,$$

which implies that $\hat{\phi}_l(y;f,\mathbf{x})$ is strictly concave in y. Thus, y is a solution to (EC.11) if and only if it satisfies the following first order optimality conditions:

$$\frac{\partial^+}{\partial y}\hat{\phi}_l(y;f,\mathbf{x}) \leq 0, \quad \text{if } 0 \leq y < k, \tag{EC.13}$$

$$\frac{\partial^-}{\partial y}\hat{\phi}_l(y;f,\mathbf{x}) \geq 0, \quad \text{if } 0 < y \leq k. \tag{EC.14}$$

It is straightforward to show that there is a unique solution given by

$$y = \left[\frac{1}{2}\xi\right]_0^k. \tag{EC.15}$$

Case (ii): $0 < \xi + f < k$. Then $\hat{\phi}_l(y;f,\mathbf{x})$ is a piecewise smooth function in y over the interval $[0,k]$. The first and second derivatives are given by

$$\frac{\partial}{\partial y}\hat{\phi}_l(y;f,\mathbf{x}) = \begin{cases} \beta\left(\frac{1}{N}\xi - \frac{N-1}{N}f - \frac{2}{N}y\right), & \text{if } 0 \leq y < \xi + f, \\ \beta\left(\xi - 2y\right), & \text{if } k \geq y > \xi + f, \end{cases}$$

$$\frac{\partial^2}{\partial y^2}\hat{\phi}_l(y;f,\mathbf{x}) = \begin{cases} -\frac{2}{N}\beta, & \text{if } 0 \leq y < \xi + f, \\ -2\beta, & \text{if } k \geq y > \xi + f, \end{cases}$$

$$< 0.$$

Moreover, we have

$$\begin{aligned} \frac{\partial^+}{\partial y}\hat{\phi}_l(y;f,\mathbf{x})\bigg|_{y=\xi+f} &= \beta\left(-\xi - 2f\right) \\ &= \frac{1}{N}\beta\left(-N\xi - 2Nf\right) \\ &\leq \frac{1}{N}\beta\left(-\xi + (N-1)f - 2Nf\right) \\ &= \frac{1}{N}\beta\left(\xi - (N-1)f - 2\xi - 2f\right) \\ &= \frac{\partial^-}{\partial y}\hat{\phi}_l(y;f,\mathbf{x})\bigg|_{y=\xi+f}, \end{aligned}$$

where the inequality follows from the fact that $\xi + f > 0$. Thus, $\hat{\phi}_l(y;f,\mathbf{x})$ is concave in y over $[0,k]$. Therefore, y is a solution to (EC.11) if and only if it satisfies the first order optimality conditions (EC.13) – (EC.14). It is straightforward to show that there is a unique solution given by

$$y = \begin{cases} 0, & \text{if } \xi \le (N-1)f, \\ \frac{1}{2}\left(\xi - (N-1)f\right), & \text{if } \xi > \max((N-1)f, -(N+1)f), \\ \xi + f, & \text{if } -2f \le \xi \le -(N+1)f, \\ \frac{1}{2}\xi, & \text{if } \xi < \min(2k, -2f), \\ k, & \text{if } \xi \ge 2k. \end{cases} \tag{EC.16}$$

Case (iii): $k \le \xi + f \le Nk$. Then $\hat{\phi}_l(y; f, \mathbf{x})$ is a smooth function in y over the interval $[0, k]$. The first and second derivatives are given by

$$\begin{aligned} \frac{\partial}{\partial y}\hat{\phi}_l(y; f, \mathbf{x}) &= \beta\left(\frac{1}{N}\xi - \frac{N-1}{N}f - \frac{2}{N}y\right), \\ \frac{\partial^2}{\partial y^2}\hat{\phi}_l(y; f, \mathbf{x}) &= -\frac{2}{N}\beta < 0, \end{aligned}$$

which implies that $\hat{\phi}_l(y; f, \mathbf{x})$ is strictly concave in y. Thus, y is a solution to (EC.11) if and only if it satisfies the first order optimality conditions (EC.13) – (EC.14). It is straightforward to show that there is a unique solution given by

$$y = \left[\frac{1}{2}\left(\xi - (N-1)f\right)\right]_0^k. \tag{EC.17}$$

Case (iv): $Nk < \xi + f < (N+1)k$. Then $\hat{\phi}_l(y; f, \mathbf{x})$ is a piecewise smooth function in y over the interval $[0, k]$. The first and second derivatives are given by

$$\begin{aligned} \frac{\partial}{\partial y}\hat{\phi}_l(y; f, \mathbf{x}) &= \begin{cases} \beta\left(\xi - 2y - (N-1)k\right), & \text{if } 0 \le y < \xi + f - Nk, \\ \beta\left(\frac{1}{N}\xi - \frac{N-1}{N}f - \frac{2}{N}y\right), & \text{if } k \ge y > \xi + f - Nk, \end{cases} \\ \frac{\partial^2}{\partial y^2}\hat{\phi}_l(y; f, \mathbf{x}) &= \begin{cases} -2\beta, & \text{if } 0 \le y < \xi + f - Nk, \\ -\frac{2}{N}\beta, & \text{if } k \ge y > \xi + f - Nk, \end{cases} \\ &< 0. \end{aligned}$$

It is straightforward to check that $\hat{\phi}_l(y; f, \mathbf{x})$ is not concave in y over the interval $[0, k]$. However, $\hat{\phi}_l(y; f, \mathbf{x})$ is piecewise concave in y. Therefore, solve the following sub-problems:

$$\sup_{y \in [0, \xi + f - (N-1)k]} \hat{\phi}_l(y; f, \mathbf{x}), \tag{EC.18}$$

and

$$\sup_{y \in [\xi + f - (N-1)k, k]} \hat{\phi}_l(y; f, \mathbf{x}). \tag{EC.19}$$

The solution of the sub-problem with the larger optimal value is the solution to (EC.11). Using the first-order optimality conditions, the unique solution to (EC.18) is given by

$$y = \left[\frac{1}{2}\left(\xi - (N-1)k\right)\right]_0^{\xi + f - Nk} =: z_1,$$

and that to (EC.19) is given by

$$y = \left[\frac{1}{2}\left(\xi - (N-1)f\right)\right]_{\xi + f - Nk}^{k} =: z_2.$$

Therefore, the solution(s) to (EC.11) are given by:

$$y = z_1, \quad \text{if } \hat{\phi}_l(z_1; f, \mathbf{x}) > \hat{\phi}_l(z_2; f, \mathbf{x}), \tag{EC.20a}$$

$$y = z_2, \quad \text{if } \hat{\phi}_l(z_1; f, \mathbf{x}) < \hat{\phi}_l(z_2; f, \mathbf{x}), \tag{EC.20b}$$

$$y = z_1 \text{ or } z_2, \quad \text{if } \hat{\phi}_l(z_1; f, \mathbf{x}) = \hat{\phi}_l(z_2; f, \mathbf{x}). \tag{EC.20c}$$

Case (v): $(N+1)k \leq \xi + f$. Then $\hat{\phi}_l(y; f, \mathbf{x})$ is a smooth function in y over the interval $[0, k]$. The first and second derivatives are given by

$$\frac{\partial}{\partial y}\hat{\phi}_l(y; f, \mathbf{x}) = \beta\left(\xi - (N-1)k - 2y\right),$$
$$\frac{\partial^2}{\partial y^2}\hat{\phi}_l(y; f, \mathbf{x}) = -2\beta < 0,$$

which implies that $\hat{\phi}_l(y; f, \mathbf{x})$ is strictly concave in y. Therefore, y is a solution to (EC.11) if and only if it satisfies the first order optimality conditions (EC.13) – (EC.14). It is straightforward to show that there is a unique solution given by

$$y = \left[\frac{1}{2}(\xi - (N-1)k)\right]_0^k. \tag{EC.21}$$

Step 3: We solve for the symmetric follower forward positions that satisfy the condition that every follower is producing at its payoff maximizing quantity. The latter gives the set of symmetric follower reactions since

$$\begin{aligned}
&\phi_l(f; f\mathbf{1}, \mathbf{x}) \geq \phi_l\left(\bar{f}; f\mathbf{1}, \mathbf{x}\right), \quad \forall \bar{f} \in \mathbb{R} \\
\Longleftrightarrow\ &\hat{\phi}_l(y_l(f; f\mathbf{1}, \mathbf{x}); f, \mathbf{x}) \geq \hat{\phi}_l\left(y_l(\bar{f}; f\mathbf{1}, \mathbf{x}); f, \mathbf{x}\right), \quad \forall \bar{f} \in \mathbb{R} \\
\Longleftrightarrow\ &\hat{\phi}_l(y; f, \mathbf{x}) \geq \hat{\phi}_l(\bar{y}; f\mathbf{1}, \mathbf{x}); f, \mathbf{x}), \quad \forall \bar{y} \in [0, k], \text{ and } y = \left[\frac{1}{N+1}(\xi + f)\right]_0^k.
\end{aligned} \tag{EC.22}$$

The first equivalence is due to (EC.10). The second equivalence is due to the fact that $y_l(f; f\mathbf{1}, \mathbf{x}) = \left[\frac{1}{N+1}(\xi + f)\right]_0^k$ and $y_l(\mathbb{R}; f\mathbf{1}, \mathbf{x}) = [0, k]$. We divide the analyses into five cases depending on the value of $\xi + f$.

Case (i): $\xi + f \leq 0$. Note that y is given by (EC.15). Therefore, the symmetric follower reactions are given by:

$$\begin{aligned}
\left[\frac{1}{2}\xi\right]_0^k = \left[\frac{1}{N+1}(\xi + f)\right]_0^k \text{ and (EC.12a) holds} \Longleftrightarrow\ &\left[\frac{1}{2}\xi\right]_0^k = 0 \text{ and (EC.12a) holds} \\
\Longleftrightarrow\ &\xi \leq 0 \text{ and } f \leq -\xi.
\end{aligned} \tag{EC.23}$$

Case (ii): $0 < \xi + f < k$. Note that y is given by (EC.16). Since $0 < \xi + f < k \implies 0 < \frac{1}{N+1}(\xi + f) < k$, the symmetric follower reactions are given by:

$$\begin{aligned}
&\frac{1}{2}(\xi - (N-1)f) = \frac{1}{N+1}(\xi + f) \text{ and } \xi > \max((N-1)f, -(N+1)f) \text{ and (EC.12b) holds} \\
&\text{or } \xi + f = \frac{1}{N+1}(\xi + f) \text{ and } -2f \leq \xi \leq -(N+1)f \text{ and (EC.12b) holds} \\
&\text{or } \frac{1}{2}\xi = \frac{1}{N+1}(\xi + f) \text{ and } \xi < \min(2k, -2f) \text{ and (EC.12b) holds}
\end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow f = \frac{N-1}{N^2+1}\xi \text{ and } \xi > \max((N-1)f, -(N+1)f) \text{ and (EC.12b) holds}\\
&\qquad \text{or } f = -\xi \text{ and } -2f \le \xi \le -(N+1)f \text{ and (EC.12b) holds}\\
&\qquad \text{or } f = \frac{N-1}{2}\xi \text{ and } \xi < \min(2k, -2f) \text{ and (EC.12b) holds}\\
&\Longleftrightarrow f = \frac{N-1}{N^2+1}\xi \text{ and } \xi > \max\left(\frac{(N-1)^2}{N^2+1}\xi, -\frac{N^2-1}{N^2+1}\xi\right) \text{ and } 0 < \xi < \frac{N^2+1}{N(N+1)}k\\
&\qquad \text{or } f = \frac{N-1}{2}\xi \text{ and } \xi < \min(2k, -(N-1)\xi) \text{ and } 0 < \frac{N+1}{2}\xi < k\\
&\Longleftrightarrow f = \frac{N-1}{N^2+1}\xi \text{ and } 0 < \xi < \frac{N^2+1}{N(N+1)}k. \qquad \text{(EC.24)}
\end{aligned}$$

The second equivalence is due to the fact that $f = -\xi \implies \xi + f = 0$. The third equivalence is due to the fact that $\xi > 0 \implies \frac{(N-1)^2}{N^2+1}\xi \ge -\frac{N^2-1}{N^2+1}\xi$ and $\frac{N+1}{2}\xi > 0 \implies 2k > -(N-1)\xi$.

Case (iii): $k \le \xi + f \le Nk$. Note that y is given by (EC.17). Therefore, the symmetric follower reactions are given by:

$$\begin{aligned}
&\left[\frac{1}{2}(\xi - (N-1)f)\right]_0^k = \left[\frac{1}{N+1}(\xi + f)\right]_0^k \text{ and (EC.12c) holds}\\
\Longleftrightarrow\ &\frac{1}{2}(\xi - (N-1)f) = \frac{1}{N+1}(\xi + f) \text{ and (EC.12c) holds and } 0 < \frac{1}{2}(\xi - (N-1)f) < k\\
\Longleftrightarrow\ &f = \frac{N-1}{N^2+1}\xi \text{ and (EC.12c) holds and } 0 < \frac{1}{2}(\xi - (N-1)f) < k\\
\Longleftrightarrow\ &f = \frac{N-1}{N^2+1}\xi \text{ and } \frac{N^2+1}{N(N+1)}k \le \xi \le \frac{N^2+1}{N+1}k \text{ and } 0 < \xi < \frac{N^2+1}{N-\frac{1}{2}}k\\
\Longleftrightarrow\ &f = \frac{N-1}{N^2+1}\xi \text{ and } \frac{N^2+1}{N(N+1)}k \le \xi \le \frac{N^2+1}{N+1}k. \qquad \text{(EC.25)}
\end{aligned}$$

The first equivalence is due to the fact that (EC.12c) $\implies 0 < \frac{1}{N+1}(\xi + f) < k$. The last equivalence is due to the fact that $\frac{N^2+1}{N(N+1)} < \frac{N^2+1}{N+1} < \frac{N^2+1}{N-\frac{1}{2}}$.

Case (iv): $Nk < \xi + f < (N+1)k$. Note that y is described by (EC.20). We show that there does not exist a symmetric follower reaction such that $y = z_1$. Suppose otherwise. By Proposition EC.1, for each $j \neq l$,

$$\begin{aligned}
y_j &= \left[\frac{1}{N}\left(\xi + f - \left[\frac{1}{2}(\xi - (N-1)k)\right]_0^{\xi+f-Nk}\right)\right]_0^k\\
&= \left[-\frac{1}{N}\left[\frac{1}{2}(-\xi - 2f - (N-1)k)\right]_{-\xi-f}^{-Nk}\right]_0^k
\end{aligned}$$

$$= \left[\left[\frac{1}{2N} \left(\xi + 2f + (N-1)k \right) \right]_k^{\frac{1}{N}(\xi+f)} \right]_0^k$$

$$= k$$

However, (EC.12d) $\implies \frac{1}{N+1}(\xi+f) < k \implies y < k = y_j$, which contradicts with the fact that a symmetric follower reaction implies symmetric productions (by Proposition EC.2).

Therefore, the symmetric follower reactions are given by:

$$\begin{aligned}
&\left[\frac{1}{2}\left(\xi-(N-1)f\right)\right]_{\xi+f-Nk}^{k} = \frac{1}{N+1}\left(\xi+f\right) \text{ and (EC.12d) holds and } \hat{\phi}_l(z_1;f,\mathbf{x}) \le \hat{\phi}_l(z_2;f,\mathbf{x}) \\
\Longleftrightarrow\ & \frac{1}{2}\left(\xi-(N-1)f\right) = \frac{1}{N+1}\left(\xi+f\right) \text{ and (EC.12d) holds and } \hat{\phi}_l(z_1;f,\mathbf{x}) \le \hat{\phi}_l(z_2;f,\mathbf{x}) \\
\Longleftrightarrow\ & f = \frac{N-1}{N^2+1}\xi \text{ and (EC.12d) holds and } \hat{\phi}_l(z_1;f,\mathbf{x}) \le \hat{\phi}_l(z_2;f,\mathbf{x}) \\
\Longleftrightarrow\ & f = \frac{N-1}{N^2+1}\xi \text{ and } \frac{N^2+1}{N+1}k < \xi < \frac{N^2+1}{N}k \text{ and } \hat{\phi}_l(z_1;f,\mathbf{x}) \le \hat{\phi}_l(z_2;f,\mathbf{x}) \\
\Longleftrightarrow\ & f = \frac{N-1}{N^2+1}\xi \text{ and } \frac{N^2+1}{N+1}k < \xi \le \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k. \qquad \text{(EC.26)}
\end{aligned}$$

The first equivalence follows from the fact that $\xi+f-Nk = \frac{1}{N+1}(\xi+f) \implies \xi+f=(N+1)k$ and $k=\frac{1}{N+1}(\xi+f) \implies \xi+f=(N+1)k$. The last equivalence follows from the following facts. First, note that

$$\begin{aligned}
z_1 &= \left[\frac{1}{2}\left(\xi-(N-1)k\right)\right]_0^{\frac{N(N+1)}{N^2+1}\xi-Nk} \\
&= \begin{cases} \frac{N(N+1)}{N^2+1}\xi - Nk, & \text{if } \frac{N^2+1}{N+1}k < \xi < \frac{N^2+1}{N}k \text{ and } \frac{1}{2}\left(\xi-(N-1)k\right) > \frac{N(N+1)}{N^2+1}\xi - Nk, \\ \frac{1}{2}\left(\xi-(N-1)k\right), & \text{if } \frac{N^2+1}{N+1}k < \xi < \frac{N^2+1}{N}k \text{ and } \frac{1}{2}\left(\xi-(N-1)k\right) \le \frac{N(N+1)}{N^2+1}\xi - Nk, \end{cases} \\
&= \begin{cases} \frac{N(N+1)}{N^2+1}\xi - Nk, & \text{if } \frac{N^2+1}{N+1}k < \xi < \frac{(N+1)(N^2+1)}{N^2+2N-1}k, \\ \frac{1}{2}\left(\xi-(N-1)k\right), & \frac{(N+1)(N^2+1)}{N^2+2N-1}k \le \xi < \frac{N^2+1}{N}k. \end{cases}
\end{aligned}$$

where the second equality is due to $\xi > \frac{N^2+1}{N+1}k > \frac{N^2-1}{N+1}k = (N-1)k$. Thus, if $\frac{N^2+1}{N+1}k < \xi < \frac{(N+1)(N^2+1)}{N^2+2N-1}k$, then

$$\hat{\phi}_l(z_1;f,\mathbf{x}) \le \hat{\phi}_l(z_2;f,\mathbf{x})$$

$$\begin{aligned}
&\iff (\xi-(N-1)k-z_1)\,z_1 \leq \frac{1}{N}\,(\xi-(N-1)f-z_2)\,z_2 \\
&\iff (k-f)\,(\xi+f-Nk) \leq \frac{1}{4N}\,(\xi-(N-1)f)^2 \\
&\iff \text{True.}
\end{aligned}$$

On the other hand, if $\frac{(N+1)(N^2+1)}{N^2+2N-1}k \leq \xi < \frac{N^2+1}{N}k$, then

$$\begin{aligned}
&\hat{\phi}_l(z_1;f,\mathbf{x}) \leq \hat{\phi}_l(z_2;f,\mathbf{x}) \\
&\iff (\xi-(N-1)k-z_1)\,z_1 \leq \frac{1}{N}\,(\xi-(N-1)f-z_2)\,z_2 \\
&\iff \frac{1}{2}\,(\xi-(N-1)k)^2 \leq \frac{1}{4N}\,(\xi-(N-1)f)^2 \\
&\iff \xi \leq \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k,
\end{aligned}$$

where $\frac{(N+1)(N^2+1)}{N^2+2N-1}k \leq \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k < \frac{N^2+1}{N}k$.

Case (v): $(N+1)k \leq \xi+f$. Note that y is given by (EC.21). Therefore, the symmetric follower reactions are given by:

$$\begin{aligned}
&\left[\frac{1}{2}\,(\xi-(N-1)k)\right]_0^k = \left[\frac{1}{N+1}\,(\xi+f)\right]_0^k \text{ and (EC.12e) holds} \\
\iff &\left[\frac{1}{2}\,(\xi-(N-1)k)\right]_0^k = k \text{ and (EC.12e) holds} \\
\iff &f \geq -\xi+(N+1)k \text{ and } \xi \geq (N+1)k. \qquad \text{(EC.27)}
\end{aligned}$$

Putting together the descriptions in (EC.23) – (EC.27) yield (EC.9).

EC.1.3. Leader Reaction Analyses

Proposition EC.4. *Fix the followers' forward positions* $\mathbf{f}=f\mathbf{1}\in\mathbb{R}^N$. *Let* $X\subseteq\mathbb{R}_+$ *denote the set of symmetric leader reactions, i.e., for each* $x\in X$ *and* $i\in M$,

$$\psi_i\,(x;x\mathbf{1},f\mathbf{1}) \geq \psi_i\,(\bar{x};x\mathbf{1},f\mathbf{1}), \quad \forall \bar{x}\in\mathbb{R}_+. \qquad \text{(EC.28)}$$

Let:

$$x_1 = \frac{1}{M+1}\,[\alpha_x]_0^\infty,$$

$$x_2 = \frac{1}{M}\left(\alpha_x - \triangle C + f\right),$$
$$x_3 = \frac{1}{M+1}\left[\alpha_x + N\left(\triangle C - f\right)\right]_0^{\infty},$$
$$x_4 = \frac{1}{M+1}\left[\alpha_x - Nk\right]_0^{\infty}.$$

Then,

$$X = \left\{ x \in \mathbb{R}_+ \;\middle|\; \begin{array}{l} x = x_1 \textit{ if } f - \triangle C < -\frac{\alpha_x}{M+1}, \\ \textit{or } x = x_2 \textit{ if } -\frac{\alpha_x}{M+1} \le f - \triangle C \le \min\left(-\frac{\alpha_x}{MN+M+1}, \max\left(\eta_4, -\alpha_x + M(N+1)k\right)\right), \\ \textit{or } x = x_3 \textit{ if } -\frac{\alpha_x}{NM+M+1} < f - \triangle C \le \max\left(\eta_3, k - (\alpha_x - Nk)\right), \\ \textit{or } x = x_4 \textit{ if } f - \triangle C \ge \begin{cases} k - (\alpha_x - Nk), & \textit{if } \alpha_x < Nk, \\ \eta_2, & \textit{if } \alpha_x \ge \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \\ \eta_1, & \textit{otherwise.} \end{cases} \end{array} \right\},$$

where

$$\eta_1 := k - \tfrac{\alpha_x - Nk}{N}\left(\tfrac{2(\sqrt{N+1}-1)}{M+1}\right),$$
$$\eta_2 := -\tfrac{1}{2}\left(\tfrac{2(\alpha_x - Nk)}{M+1} + Nk\right)\left(1 - \sqrt{1 - \left(\frac{\frac{2(\alpha_x - Nk)}{M+1}}{\frac{2(\alpha_x - Nk)}{M+1} + Nk}\right)^2}\right),$$
$$\eta_3 := k - \tfrac{\alpha_x - Nk}{N}\left(\tfrac{2(\sqrt{N+1}-1)}{2+(M-1)\sqrt{N+1}}\right),$$
$$\eta_4 := -\tfrac{1}{2}\left(\tfrac{2(\alpha_x - MNk)}{M+1} + \left(\tfrac{2M}{M+1}\right)^2 Nk\right)\left(1 - \sqrt{1 - \left(\frac{\frac{2(\alpha_x - MNk)}{M+1}}{\frac{2(\alpha_x - MNk)}{M+1} + \left(\frac{2M}{M+1}\right)^2 Nk}\right)^2}\right).$$

Moreover, for each $x \in X$,

$$y_j(f\mathbf{1}, x\mathbf{1}) = 0 \iff x = x_1 \textit{ or } x_2,$$
$$0 < y_j(f\mathbf{1}, x\mathbf{1}) < k \iff x = x_3,$$
$$y_j(f\mathbf{1}, x\mathbf{1}) = k \iff x = x_4.$$

Proof. The proof proceeds in three steps. In step 1, we solve for a leader's payoff maximizing production quantity given that all other leaders produce equal quantities. In step 2, we solve for the symmetric leader productions that satisfy the condition that every leader is producing at its payoff maximizing quantity. The latter gives the set of symmetric leader reactions. In step 3, we explain how the solutions obtained in step 2 is equivalent to X.

Step 1: Fix a leader $l \in M$ and suppose $x_i = x$ for every $i \neq l$. We solve for the solution to

$$\sup_{x_l \in \mathbb{R}_+} \psi_l \left(x_l; x\mathbf{1}, f\mathbf{1}\right). \tag{EC.29}$$

Substituting for the demand function yields

$$\psi_l \left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \beta \left(\alpha_x - x_l - (M-1)x - \sum_{j=1}^{N} y_j(f\mathbf{1}, x_l, x\mathbf{1})\right) x_l$$

where the follower productions are given by

$$\begin{aligned}
&\sum_{j=1}^{N} y_j(f\mathbf{1}, x_l, x\mathbf{1}) \\
&= N\left[\frac{1}{N+1}\left(\alpha_y + f - x_l - (M-1)x\right)\right]_0^k \\
&= \begin{cases} 0, & \text{if (EC.30a) holds,} \\ \begin{cases} 0, & \text{if } \alpha_y + f - x_l - (M-1)x \leq 0, \\ \frac{N}{N+1}\left(\alpha_y + f - x_l - (M-1)x\right), & \text{otherwise,} \end{cases} & \text{if (EC.30b) holds,} \\ \begin{cases} 0, & \text{if } \alpha_y + f - x_l - (M-1)x \leq 0, \\ k, & \text{if } \alpha_y + f - x_l - (M-1)x \geq (N+1)k, \\ \frac{N}{N+1}\left(\alpha_y + f - x_l - (M-1)x\right), & \text{otherwise,} \end{cases} & \text{if (EC.30c) holds,} \end{cases}
\end{aligned}$$

where the second equality is due to the fact that $x_l, x \geq 0$ and the three cases (EC.30a) – (EC.30c) are defined by

$$\alpha_y + f - (M-1)x \leq 0, \tag{EC.30a}$$

$$0 < \alpha_y + f - (M-1)x \leq (N+1)k, \tag{EC.30b}$$

$$(N+1)k < \alpha_y + f - (M-1)x. \tag{EC.30c}$$

We analyze each case separately.

Case (i): $\alpha_y + f - (M-1)x < 0$. We obtain

$$\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \beta\left(\alpha_x - x_l - (M-1)x\right)x_l.$$

Therefore, $\psi_l(x_l; x\mathbf{1}, f\mathbf{1})$ is a smooth function in x_l over $\mathbb{R}_+$. The first and second derivatives are given by

$$\frac{\partial}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \beta\left(\alpha_x - (M-1)x - 2x_l\right),$$
$$\frac{\partial^2}{\partial x_l^2}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) = -2\beta < 0,$$

which implies that $\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is strictly concave in x_l. Therefore, x_l is a solution to (EC.29) if and only if it satisfies the following first order optimality conditions:

$$\frac{\partial^+}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) \le 0, \quad \text{if } 0 \le x_l, \tag{EC.31}$$
$$\frac{\partial^-}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) \ge 0, \quad \text{if } 0 < x_l. \tag{EC.32}$$

It is straightforward to show that there is a unique solution is given by

$$x_l = \left[\frac{1}{2}\left(\alpha_x - (M-1)x\right)\right]_0^\infty. \tag{EC.33}$$

Case (ii): $0 \le \alpha_y + f - (M-1)x < (N+1)k$. We obtain

$$\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \begin{cases} \beta\left(\alpha_x - x_l - (M-1)x\right)x_l, & \text{if } x_l \ge \alpha_y + f - (M-1)x, \\ \beta\left(\frac{1}{N+1}\alpha_x + \frac{N}{N+1}\left(\triangle C - f\right) - \frac{M-1}{N+1}x - \frac{1}{N+1}x_l\right)x_l, & \text{otherwise.} \end{cases}$$

Therefore, $\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is a piecewise smooth function in x_l over $\mathbb{R}_+$. The first and second derivatives are given by

$$\frac{\partial}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \begin{cases} \beta\left(\alpha_x - (M-1)x - 2x_l\right), & \text{if } x_l > \alpha_y + f - (M-1)x, \\ \frac{\beta}{N+1}\left(\alpha_x + N\left(\triangle C - f\right) - (M-1)x - 2x_l\right), & \text{otherwise,} \end{cases}$$

$$\frac{\partial^2}{\partial x_l^2}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) = \begin{cases} -2\beta, & \text{if } x_l > \alpha_y + f - (M-1)x, \\ -\frac{2}{N+1}\beta, & \text{otherwise,} \end{cases}$$
$$< 0.$$

Moreover, we have

$$\begin{aligned}
&\frac{\partial^-}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\Big|_{x_l=\alpha_y+f-(M-1)x} \\
&= \beta\left(\frac{1}{N+1}\alpha_x + \frac{N}{N+1}\left(\triangle C + f\right) - \frac{M-1}{N+1}x - \frac{2}{N+1}\left(\alpha_y + f - (M-1)x\right)\right) \\
&= \beta\left(-\alpha_x + 2\left(\triangle C - f\right) + \frac{N}{N+1}\left(\alpha_y + f\right) + \frac{M-1}{N+1}x\right) \\
&\geq \beta\left(-\alpha_x + 2\left(\triangle C - f\right) + \left(M-1\right)x\right) \\
&= \frac{\partial^+}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\Big|_{x_l=\alpha_y+f-(M-1)x},
\end{aligned}$$

where the inequality follows from (EC.30b). Therefore, $\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is concave in x_l over $\mathbb{R}_+$. Therefore, x_l is a solution to (EC.29) if and only if it satisfies the first order optimality conditions (EC.31) – (EC.32). It is straightforward to show that there is a unique solution given by

$$x_l = \begin{cases} 0, & \text{if (EC.35a) holds,} \\ \frac{1}{2}\left(\alpha_x + N\left(\triangle C - f\right) - \left(M-1\right)x\right), & \text{if (EC.35b) holds,} \\ \alpha_x - \triangle C + f - (M-1)x, & \text{if (EC.35c) holds,} \\ \frac{1}{2}\left(\alpha_x - (M-1)x\right), & \text{if (EC.35d) holds,} \end{cases} \tag{EC.34}$$

where the cases (EC.35a) – (EC.35d) are defined by:

$$\alpha_x + N\left(\triangle C - f\right) \leq (M-1)x, \tag{EC.35a}$$

$$(M-1)x < \min\left(\alpha_x + N(\triangle C - f), \alpha_x - (N+2)(\triangle C - f)\right), \tag{EC.35b}$$

$$\alpha_x - (N+2)(\triangle C - f) \leq (M-1)x \leq \alpha_x - 2(\triangle C - f), \tag{EC.35c}$$

$$\alpha_x - 2(\triangle C - f) < (M-1)x. \tag{EC.35d}$$

Case (iii): $(N+1)k \leq \alpha_y + f - (M-1)x$. We obtain

$$\begin{aligned}
&\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right) \\
&= \begin{cases} \beta\left(\alpha_x - x_l - (M-1)x\right)x_l, & \text{if } x_l \geq \alpha_y + f - (M-1)x, \\ \beta\left(\alpha_x - x_l - (M-1)x - Nk\right)x_l, & \text{if } x_l \leq \alpha_y + f - (M-1)x - (N+1)k, \\ \beta\left(\frac{1}{N+1}\alpha_x + \frac{N}{N+1}\left(\triangle C - f\right) - \frac{M-1}{N+1}x - \frac{1}{N+1}x_l\right)x_l, & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, $\phi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is a piecewise smooth function in x_l over $\mathbb{R}_+$. The first and second derivatives are given by

$$
\begin{aligned}
&\frac{\partial}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\\
&=\begin{cases}
\beta\left(\alpha_x-(M-1)x-2x_l\right), & \text{if } x_l>\alpha_y+f-(M-1)x,\\
\beta\left(\alpha_x-(M-1)x-Nk-2x_l\right), & \text{if } x_l<\alpha_y+f-(M-1)x-(N+1)k,\\
\frac{\beta}{N+1}\left(\alpha_x+N\left(\triangle C-f\right)-(M-1)x-2x_l\right), & \text{otherwise,}
\end{cases}\\
&\frac{\partial^2}{\partial x_l^2}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\\
&=\begin{cases}
-2\beta, & \text{if } x_l>\alpha_y+f-(M-1)x,\\
-2\beta, & \text{if } x_l<\alpha_y+f-(M-1)x-(N+1k,\\
-\frac{2}{N+1}\beta, & \text{otherwise,}
\end{cases}\\
&<0.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\frac{\partial^-}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\Big|_{x_l=\alpha_y+f-(M-1)x}\\
&=\beta\left(\frac{1}{N+1}\alpha_x+\frac{N}{N+1}\left(\triangle C-f\right)-\frac{M-1}{N+1}x-\frac{2}{N+1}\left(\alpha_y+f-(M-1)x\right)\right)\\
&=\beta\left(-\alpha_x+2(\triangle C-f)+\frac{N}{N+1}(\alpha_x-\triangle C+f)+\frac{M-1}{N+1}x\right)\\
&>\beta\left(-\alpha_x+2(\triangle C-f)+(M-1)x\right)\\
&=\frac{\partial^+}{\partial x_l}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)\Big|_{x_l=\alpha_y+f-(M-1)x}.
\end{aligned}
$$

Therefore, $\phi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is concave in x_l over $[\alpha_y+f-(M-1)x-(N+1)k, \infty)$. However, it is straightforward to check that $\phi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ has a non-concave kink at $x_l=\alpha_y+f-(M-1)x-(N+1)k$, and therefore $\phi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right)$ is not concave in x_l over $\mathbb{R}_+$. Therefore, solve the following sub-problems:

$$
\sup_{x_l\in\left[0,\alpha_y+f-(M-1)x-(N+1)k\right]}\psi_l\left(x_l; x\mathbf{1}, f\mathbf{1}\right), \tag{EC.36}
$$

and

$$\sup_{x_l \in \left[\alpha_y + f - (M-1)x - (N+1)k, \infty\right)} \psi_l \left(x_l; x\mathbf{1}, f\mathbf{1}\right). \tag{EC.37}$$

The solution of the sub-problem with the larger optimal value is the solution to (EC.29). Using the first order optimality conditions, the unique solution to (EC.36) is given by

$$x_l = \left[\frac{1}{2}\left(\alpha_x - (M-1)x - Nk\right)\right]_0^{\alpha_y + f - (M-1)x - (N+1)k} =: z_1,$$

and that to (EC.37) is given by

$$x_l = \begin{cases} \alpha_y + f - (M-1)x - (N+1)k, & \text{if (EC.38a) holds,} \\ \frac{1}{2}\left(\alpha_x + N(\triangle C - f) - (M-1)x\right), & \text{if (EC.38b) holds,} \\ \alpha_y + f - (M-1)x, & \text{if (EC.38c) holds,} \\ \frac{1}{2}\left(\alpha_x - (M-1)x\right), & \text{if (EC.38d) holds,} \end{cases} =: z_2.$$

where the cases (EC.38a) – (EC.38d) are defined by:

$$(M-1)x \le \alpha_x - (N+2)(\triangle C - f) - 2(N+1)k, \tag{EC.38a}$$

$$\alpha_x - (N+2)(\triangle C - f) - 2(N+1)k < (M-1)x < \alpha_x - (N+2)(\triangle C - f), \tag{EC.38b}$$

$$\alpha_x - (N+2)(\triangle C - f) \le (M-1)x \le \alpha_x - 2(\triangle C - f), \tag{EC.38c}$$

$$\alpha_x - 2(\triangle C - f) < (M-1)x. \tag{EC.38d}$$

Therefore, the solution(s) to (EC.29) are given by:

$$x_l = z_1, \quad \text{if } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) > \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right), \tag{EC.39a}$$

$$x_l = z_2, \quad \text{if } \phi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) > \phi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right), \tag{EC.39b}$$

$$x_l = z_1 \text{ or } z_2, \quad \text{if } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) = \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right). \tag{EC.39c}$$

Step 2: We solve for the symmetric leader productions that satisfy the condition that every leader is producing at its payoff maximizing quantity. We divide the analyses into three cases depending on the value of $\alpha_y + f - (M-1)x$.

Case (i): $\alpha_y + f - (M-1)x < 0$. Note that x_l is given by (EC.33). Therefore, the symmetric leader reactions are given by:

$$\begin{aligned}
& x = 0 \text{ and } \alpha_x < (M-1)x \text{ and (EC.30a) holds} \\
& \text{or } x = \frac{1}{2}\left(\alpha_x - (M-1)x\right) \text{ and } \alpha_x \geq (M-1)x \text{ and (EC.30a) holds} \\
\Longleftrightarrow\; & x = 0 \text{ and } \alpha_x < 0 \text{ and (EC.30a) holds} \\
& \text{or } x = \frac{1}{M+1}\alpha_x \text{ and } \alpha_x \geq 0 \text{ and (EC.30a) holds.}
\end{aligned}$$

Case (ii): $0 \leq \alpha_y + f - (M-1)x < (N+1)k$. Note that x_l is given by (EC.34). Therefore, the symmetric leader reactions are given by:

$$\begin{aligned}
& x = 0 \text{ and (EC.35a) and (EC.30b) holds} \\
& \text{or } x = \frac{1}{2}\left((N+1)\alpha_x - N(\alpha_y + f) - (M-1)x\right) \text{ and (EC.35b) and (EC.30b) holds} \\
& \text{or } x = \alpha_y + f - (M-1)x \text{ and (EC.35c) and (EC.30b) holds} \\
& \text{or } x = \frac{1}{2}\left(\alpha_x - (M-1)x\right) \text{ and (EC.35d) and (EC.30b) holds} \\
\Longleftrightarrow\; & x = 0 \text{ and } \frac{\alpha_x}{N} \leq f - \triangle C \text{ and (EC.30b) holds} \\
& \text{or } x = \frac{1}{M+1}\left(\alpha_x + N(\triangle C - f)\right) \text{ and } -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{\alpha_x}{N} \text{ and (EC.30b) holds} \\
& \text{or } x = \frac{1}{M}\left(\alpha_x - (\triangle C - f)\right) \text{ and } -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq -\frac{\alpha_x}{NM+M+1} \text{ and (EC.30b) holds} \\
& \text{or } x = \frac{1}{M+1}\alpha_x \text{ and } f - \triangle C < -\frac{\alpha_x}{M+1} \text{ and (EC.30b) holds.}
\end{aligned}$$

Case (iii): $(N+1)k \leq \alpha_y + f - (M-1)x$. Note that x_l is described by (EC.39). Therefore, the symmetric leader reactions are given by

$$x = z_1 \text{ and (EC.30c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) > \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \tag{EC.40}$$

$$\text{or } x = z_2 \text{ and (EC.30c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) < \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \tag{EC.41}$$

$$\text{or } x = z_1 \text{ or } z_2 \text{ and (EC.30c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) = \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right). \tag{EC.42}$$

We analyze the cases $x = z_1$ and $x = z_2$ separately.

Suppose $x = z_1$ is a symmetric leader reaction. Since

$$
\begin{aligned}
&\frac{\partial^-}{\partial x_l}\psi_l(x_l;x\mathbf{1},f\mathbf{1})\Big|_{x_l=\alpha_y+f-(M-1)x-(N+1)k} \\
&= \beta\left(-\alpha_x + 2(\triangle C - f) + (M-1)x + (N+2)k\right) \\
&\le \frac{\beta}{N+1}\left(-\alpha_x + (N+2)(\triangle C - f) + (M-1)x + 2(N+1)k\right) \\
&= \frac{\partial^+}{\partial x_l}\psi_l(x_l;x\mathbf{1},f\mathbf{1})\Big|_{x_l=\alpha_y+f-(M-1)x-(N+1)k},
\end{aligned}
$$

we infer that $x < \alpha_y + f - (M-1)x - (N+1)k$. Therefore, we obtain

$$
\begin{aligned}
& x = z_1 \\
\iff & x = 0 \text{ and (EC.30c) holds and } \psi_l(z_1;x\mathbf{1},f\mathbf{1}) \ge \psi_l(z_2;x\mathbf{1},f\mathbf{1}) \text{ and } \alpha_x - (M-1)x - Nk \le 0 \\
& \text{or } x = \frac{1}{2}\left(\alpha_x - (M-1)x - Nk\right) \text{ and (EC.30c) holds and } \psi_l(z_1;x\mathbf{1},f\mathbf{1}) \ge \psi_l(z_2;x\mathbf{1},f\mathbf{1}) \\
& \quad \text{and } 0 < \frac{1}{2}\left(\alpha_x - (M-1)x - Nk\right) < \alpha_y + f - (M-1)x - (N+1)k \\
\iff & x = 0 \text{ and (EC.30c) holds and } \alpha_x - Nk \le 0 \\
& \text{or } x = \frac{1}{M+1}\left(\alpha_x - Nk\right) \text{ and (EC.30c) holds and } \psi_l(z_1;x\mathbf{1},f\mathbf{1}) \ge \psi_l(z_2;x\mathbf{1},f\mathbf{1}) \\
& \quad \text{and } \alpha_x - Nk > 0 \text{ and } f - \triangle C > -\frac{1}{M+1}\left(\alpha_x - Nk\right) + k
\end{aligned}
$$

The second equivalence follows from solving for x in the equations, and the fact that in the case $x = 0$, the inequalities (EC.30c) and $\alpha_x - Nk \le 0 \implies \psi_l(z_1;x\mathbf{1},f\mathbf{1}) \ge \psi_l(z_2;x\mathbf{1},f\mathbf{1})$.

Suppose $x = z_2$. Then, using the same arguments in (EC.40), we infer that $x < \alpha_y + f - (M-1)x - (N+1)k$. Therefore, we obtain

$$
\begin{aligned}
& x = z_2 \\
\iff & x = \frac{1}{2}\left(\alpha_x + N(\triangle C - f) - (M-1)x\right) \text{ and (EC.30c) and (EC.38b) holds} \\
& \quad \text{and } \psi_l(z_1;x\mathbf{1},f\mathbf{1}) \le \psi_l(z_2;x\mathbf{1},f\mathbf{1}) \\
& \text{or } x = \alpha_x + (f - \triangle C) - (M-1)x \text{ and (EC.30c) and (EC.38c) holds} \\
& \quad \text{and } \psi_l(z_1;x\mathbf{1},f\mathbf{1}) \le \psi_l(z_2;x\mathbf{1},f\mathbf{1})
\end{aligned}
$$

$$
\begin{aligned}
&\text{or } x = \frac{1}{2}\left(\alpha_x - (M-1)x\right) \text{ and (EC.30c) and (EC.38d) holds} \\
&\quad \text{and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \le \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
\iff & x = \frac{1}{M+1}\left(\alpha_x + N(\triangle C - f)\right) \text{ and (EC.30c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \le \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
&\quad \text{and } -\frac{1}{NM+M+1}\alpha_x < f - \triangle C < \frac{1}{NM+M+1}\left(-\alpha_x + (M+1)(N+1)k\right) \\
&\text{or } x = \frac{1}{M}\left(\alpha_x + (f - \triangle C)\right) \text{ and (EC.30c) holds and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \le \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \\
&\quad \text{and } -\frac{1}{M+1}\alpha_x \le f - \triangle C \le -\frac{1}{NM+M+1}\alpha_x \\
&\text{or } x = \frac{1}{M+1}\alpha_x \text{ and (EC.30c) holds and and } f - \triangle C < -\frac{1}{M+1}\alpha_x
\end{aligned}
$$

The second equivalence follows from solving for x in the equations, and the fact that in the case $x = \frac{1}{M+1}\alpha_x$, the inequalities (EC.30c) and $f - \triangle C < -\frac{1}{M+1}\alpha_x \implies \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \le \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right)$.

Step 3: We explain how the solutions obtained in step 2 is equivalent to X. Observe that step 2 obtains five cases for x:

$$
\begin{aligned}
x &= \frac{1}{M+1}\alpha_x, \\
x &= \frac{1}{M}\left(\alpha_x - (\triangle C - f)\right), \\
x &= \frac{1}{M+1}\left(\alpha_x + N(\triangle C - f)\right), \\
x &= \frac{1}{M+1}\left(\alpha_x - Nk\right), \\
x &= 0.
\end{aligned}
$$

We analyze each case separately.

Case (i): $x = \frac{1}{M+1}\alpha_x$. This case is characterized by

$$
\begin{aligned}
&\alpha_x \ge 0 \text{ and (EC.30a) holds} \\
&\text{or } f - \triangle C < -\frac{\alpha_x}{M+1} \text{ and (EC.30b) holds} \\
&\text{or } f - \triangle C < -\frac{\alpha_x}{M+1} \text{ and (EC.30c) holds} \\
\iff &\alpha_x \ge 0 \text{ and } f - \triangle C < -\frac{\alpha_x}{M+1}.
\end{aligned}
$$

The equivalence is due to the following facts. First, (EC.30b) $\implies \alpha_x \geq 0$ and (EC.30c) $\implies \alpha_x \geq 0$. Second, (EC.30a) and $\alpha_x \geq 0 \implies f - \triangle C < -\frac{1}{M+1}\alpha_x$. Third, (EC.30a), (EC.30b), (EC.30c) $\implies$ True.

Case (ii): $x = \frac{1}{M}\left(\alpha_x - (\triangle C - f)\right)$. This case is characterized by

$$
\begin{aligned}
& -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq -\frac{\alpha_x}{NM+M+1} \text{ and (EC.30b) holds} \\
& \text{or } -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq -\frac{\alpha_x}{NM+M+1} \\
& \quad \text{and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \text{ and (EC.30c) holds} \\
\iff & -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq \min\left(-\frac{\alpha_x}{NM+M+1}, -\alpha_x + M(N+1)k\right) \\
& \text{or } -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq -\frac{\alpha_x}{NM+M+1} \\
& \quad \text{and } f - \triangle C \leq \eta_4 \text{ and } f - \triangle C > -\alpha_x + M(N+1)k \\
\iff & -\frac{\alpha_x}{M+1} \leq f - \triangle C \leq \min\left(-\frac{\alpha_x}{MN+M+1}, \max\left(\eta_4, -\alpha_x + M(N+1)\right)\right).
\end{aligned}
$$

The first equivalence is due to the following facts. First, (EC.30b) $\iff -\alpha_x < f - \triangle C \leq -\alpha_x + M(N+1)k$. Second, $\psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \iff f - \triangle C \leq \eta_4$. Third, (EC.30c) $\iff f - \triangle C > -\alpha_x + M(N+1)k$. The second equivalence follows from combining the inequalities.

Case (iii): $x = \frac{1}{M+1}\left(\alpha_x + N(\triangle C - f)\right)$. This case is characterized by

$$
\begin{aligned}
& -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{\alpha_x}{N} \text{ and (EC.30b) holds} \\
& \text{or } -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{1}{NM+M+1}\left(-\alpha_x + (M+1)(N+1)k\right) \\
& \quad \text{and } \psi_l\left(z_1; x\mathbf{1}, f\mathbf{1}\right) \leq \psi_l\left(z_2; x\mathbf{1}, f\mathbf{1}\right) \text{ and (EC.30c) holds} \\
\iff & -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{\alpha_x}{N} \text{ and } f - \triangle C \leq \frac{-2\alpha_x + (M+1)(N+1)k}{M+1+N(M-1)} \\
& \text{or } -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{1}{NM+M+1}\left(-\alpha_x + (M+1)(N+1)k\right) \\
& \quad \text{and } f - \triangle C \leq \eta_3 \text{ and } f - \triangle C > \frac{-2\alpha_x + (M+1)(N+1)k}{M+1+N(M-1)} \\
\iff & \alpha_x < Nk \text{ and } -\frac{\alpha_x}{NM+M+1} < f - \triangle C < \frac{\alpha_x}{N} \\
& \text{or } \alpha_x \geq Nk \text{ and } -\frac{\alpha_x}{NM+M+1} < f - \triangle C \leq \eta_3.
\end{aligned}
$$

The first equivalence is due to the following facts. First, (EC.30b) $\iff -\frac{2\alpha_x}{M+1+N(M-1)} < f - \triangle C \leq \frac{-2\alpha_x+(M+1)(N+1)k}{M+1+N(M-1)}$. Second, (EC.30c) $\iff f - \triangle C > \frac{-2\alpha_x+(M+1)(N+1)k}{M+1+N(M-1)}$. Third, $\psi_l(z_1; x\mathbf{1}, f\mathbf{1}) \leq \psi_l(z_2; x\mathbf{1}, f\mathbf{1}) \iff f - \triangle C \leq \eta_3$. The second equivalence is due to the following facts. First, $\frac{\alpha_x}{N} \leq \frac{-2\alpha_x+(M+1)(N+1)k}{M+1+N(M-1)} \iff \alpha_x \leq Nk$. Second, $\alpha_x \geq Nk \implies \eta_3 < \frac{1}{NM+M+1}(-\alpha_x + (M+1)(N+1)k)$.

Case (iv): $x = \frac{1}{M+1}(\alpha_x - Nk)$. This case is characterized by

$$\begin{aligned}
&\alpha_x - Nk > 0 \text{ and } f - \triangle C > -\frac{1}{M+1}(\alpha_x - Nk) + k \\
&\quad \text{and } \psi_l(z_1; x\mathbf{1}, f\mathbf{1}) \geq \psi_l(z_2; x\mathbf{1}, f\mathbf{1}) \text{ and (EC.30c) holds} \\
\iff &\alpha_x - Nk > 0 \text{ and } f - \triangle C > -\frac{1}{M+1}(\alpha_x - Nk) + k \text{ and } \psi_l(z_1; x\mathbf{1}, f\mathbf{1}) \geq \psi_l(z_2; x\mathbf{1}, f\mathbf{1}) \\
\iff &Nk < \alpha_x < \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk \text{ and } f - \triangle C \geq \eta_1 \\
&\text{or } \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk \leq \alpha_x \text{ and } f - \triangle C \geq \eta_2.
\end{aligned}$$

The first equivalence is due to the fact that (EC.30c) $\implies f - \triangle C > -\frac{1}{M+1}(\alpha_x - Nk) + k$. The second equivalence is due to the fact that $\psi_l(z_1; x\mathbf{1}, f\mathbf{1}) \geq \psi_l(z_2; x\mathbf{1}, f\mathbf{1}) \iff Nk < \alpha_x < \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk$ and $f - \triangle C \geq \eta_1$ or $\left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk \leq \alpha_x$ and $f - \triangle C \geq \eta_2$.

Case (v): $x = 0$. This case is characterized by

$$\begin{aligned}
&\alpha_x < 0 \text{ and (EC.30a)} \\
&\text{or } \frac{\alpha_x}{N} \leq f - \triangle C \text{ and (EC.30b)} \\
&\text{or } \alpha_x - Nk \leq 0 \text{ and (EC.30c)} \\
\iff &\alpha_x < 0 \\
&\text{or } \alpha_x \geq 0 \text{ and } \alpha_x + N(\triangle C - f) \leq 0 \text{ and } 0 \leq \alpha_x + (f - \triangle C) < (N+1)k \\
&\text{or } \alpha_x \geq 0 \text{ and } \alpha_x - Nk \leq 0 \text{ and } (N+1)k \leq \alpha_x + (f - \triangle C).
\end{aligned}$$

The equivalence is due to the following facts. First, (EC.30b) and $\alpha_x < 0 \implies \alpha_x + N(\triangle C - f) < 0$. Second, (EC.30c) and $\alpha_x < 0 \implies \alpha_x - Nk < 0$.

EC.1.4. Forward Market Equilibrium

THEOREM EC.1. *Suppose $\alpha_x > 0$. Let $Q \subseteq \mathbb{R}\times\mathbb{R}_+$ denote the set of all symmetric Nash equilibria, i.e., $(f,x)\in Q$ if $(f\mathbf{1}, x\mathbf{1})$ is a Nash equilibrium of the forward market. Let:*

$$Q_1 := \left\{ (f,x)\in\mathbb{R}\times\mathbb{R}_+ \,\middle|\, \begin{array}{l} x = \frac{1}{M+1}\alpha_x \\ f < \triangle C - \frac{1}{M+1}\alpha_x \end{array} \right\},$$

$$Q_2 := \left\{ (f,x)\in\mathbb{R}\times\mathbb{R}_+ \,\middle|\, \begin{array}{l} x = \frac{1}{M}\left(\alpha_x - (\triangle C - f)\right) \\ \max\left(0, \triangle C - \frac{\alpha_x}{M+1}\right) \le f \le \triangle C + \min\left(-\frac{\alpha_x}{MN+M+1}, \max\left(\eta_4, -\alpha_x + M(N+1)k\right)\right) \end{array} \right\},$$

$$Q_3 := \left\{ (f,x)\in\mathbb{R}\times\mathbb{R}_+ \,\middle|\, \begin{array}{l} x = \frac{N+1}{N^2+MN+M+1}\left(\alpha_x + N^2\triangle C\right) \\ f = \frac{N-1}{N^2+MN+M+1}\left(\alpha_x - (MN+M+1)\triangle C\right) \end{array} \right\},$$

$$Q_4 := \left\{ (f,x)\in\mathbb{R}\times\mathbb{R}_+ \,\middle|\, \begin{array}{l} x = \frac{1}{M+1}\left(\alpha_x - Nk\right) \\ f \ge \triangle C + \begin{cases} \eta_1, & \text{if } Nk < \alpha_x \le \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right)Nk, \\ \eta_2, & \text{if } \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right)Nk < \alpha_x. \end{cases} \end{array} \right\}.$$

where η_1, η_2, η_4 are as defined in Proposition EC.4. Then,

$$Q = \left\{ (f,x)\in\mathbb{R}\times\mathbb{R}_+ \,\middle|\, \begin{array}{l} (f,x)\in Q_1 \text{ if } \alpha_x \le (M+1)\triangle C, \\ \text{or } (f,x)\in Q_2 \text{ if } \alpha_x \le \min\left((MN+M+1)\triangle C, \max\left(\zeta_1, \triangle C + M(N+1)k\right)\right), \\ \text{or } (f,x)\in Q_3 \text{ if } (MN+M+1)\triangle C < \alpha_x \le \zeta_2, \\ \text{or } (f,x)\in Q_4 \text{ if } (M+1)(\triangle C + k) + Nk \le \alpha_x. \end{array} \right\},$$

where

$$\zeta_1 = MNk + (M+1)\triangle C + 2M\sqrt{Nk\triangle C}, \tag{EC.43}$$

$$\zeta_2 = (MN+M+1)\triangle C + \frac{N^2+NM+M+1}{N(N+1)+2(1-\sqrt{N+1})}\left(Nk - (\sqrt{N+1}-1)^2\triangle C\right) \tag{EC.44}$$

Moreover, for each $(f,x)\in Q$,

$$y_j(f\mathbf{1}, x\mathbf{1}) = 0 \iff (f,x)\in Q_1\cup Q_2,$$

$$0 < y_j(f\mathbf{1}, x\mathbf{1}) < k \iff (f,x)\in Q_3,$$

$$y_j(f\mathbf{1}, x\mathbf{1}) = k \iff (f,x)\in Q_4.$$

Proof. The symmetric equilibria are given by the intersection of the follower and leader reactions obtained in Propositions EC.3 and EC.4. We divide the analyses into three separate cases depending on the value of the follower productions $y_j(f\mathbf{1}, x\mathbf{1})$.

Case (i): $y_j(f\mathbf{1}, x\mathbf{1}) = 0$. Using Propositions EC.3 and EC.4, we infer that (f,x) is a symmetric equilibrium with $y_j(f\mathbf{1}, x\mathbf{1}) = 0$ if and only if

$$f \le -(\alpha_x - \triangle C - Mx), \tag{EC.45a}$$

$$0 \ge \alpha_x - \triangle C - Mx, \tag{EC.45b}$$

$$x = \begin{cases} \frac{1}{M+1}[\alpha_x]_0^\infty, & \text{if } f - \triangle C < -\frac{\alpha_x}{M+1}, \\ \frac{1}{M}(\alpha_x - \triangle C + f), & \text{if } -\frac{\alpha_x}{M+1} \le f - \triangle C \le \min\left(-\frac{\alpha_x}{MN+M+1}, \max(\eta_4, -\alpha_x + M(N+1)k)\right). \end{cases} \tag{EC.45c}$$

Suppose $x = \frac{1}{M+1}[\alpha_x]_0^\infty$. Since $\alpha_x > 0$, we infer that $x = \frac{1}{M+1}\alpha_x$. Substituting into (EC.45a) and (EC.45b) yields

$$\text{(EC.45a)} \iff f < \triangle C - \frac{\alpha_x}{M+1},$$
$$\text{(EC.45b)} \iff \alpha_x \le (M+1)\triangle C.$$

The above inequalities, together with (EC.45c), imply that (f,x) satisfies (EC.45) with $x = \frac{1}{M+1}\alpha_x$, if and only if $(f,x) \in Q_1$ and $\alpha_x \le (M+1)\triangle C$.

Suppose $x = \frac{1}{M}(\alpha_x - \triangle C + f)$. Substituting into (EC.45a) and (EC.45b) yields

$$\text{(EC.45a)} \iff f \le f \iff \text{True},$$
$$\text{(EC.45b)} \iff f \ge 0.$$

Therefore, there exists (f,x) that satisfies (EC.45) with $x = \frac{1}{M}(\alpha_x - \triangle C + f)$ if and only if

$$\left[\max\left(0, \triangle C - \frac{\alpha_x}{M+1}\right), \triangle C + \min\left(-\frac{\alpha_x}{MN+M+1}, \max(\eta_4, -\alpha_x + M(N+1)k)\right)\right] \neq \varnothing$$
$$\iff 0 \le \triangle C + \min\left(-\frac{\alpha_x}{MN+M+1}, \max(\eta_4, -\alpha_x + M(N+1)k)\right)$$
$$\iff \alpha_x \le (MN+M+1)\triangle C \text{ and } 0 \le \triangle C + \max(\eta_4, -\alpha_x + M(N+1)k)$$
$$\iff \alpha_x \le (MN+M+1)\triangle C \text{ and } \alpha_x \le \max(\zeta_1, \triangle C + M(N+1)k).$$

Therefore, (f,x) satisfies (EC.45) with $x = \frac{1}{M}\left(\alpha_x - \triangle C + f\right)$, if and only if $(f,x) \in Q_2$ and $\alpha_x \leq \min\left((MN+M+1)\triangle C, \max\left(\zeta_1, \triangle C + M(N+1)k\right)\right)$.

Case (ii): $0 \leq y_j(f\mathbf{1}, x\mathbf{1}) \leq k$. Using Propositions EC.3 and EC.4, we infer that (f,x) is a symmetric equilibrium if and only if

$$f = \frac{N-1}{N^2+1}\left(\alpha_x - \triangle C - Mx\right), \tag{EC.46a}$$

$$0 \leq \alpha_x - \triangle C - Mx \leq \xi_1, \tag{EC.46b}$$

$$x = \frac{1}{M+1}\left[\alpha_x + N(\triangle C - f)\right]_0^\infty, \tag{EC.46c}$$

$$-\frac{\alpha_x}{MN+M+1} < f - \triangle C \leq \max\left(k - (\alpha_x - Nk), \eta_3\right). \tag{EC.46d}$$

We show that $x > 0$. Suppose otherwise. Substituting into (EC.46a) implies that $f = \frac{N-1}{N^2+1}(\alpha_x - \triangle C)$. Substituting further into (EC.46c) yields

$$\alpha_x + N\left(\triangle C - \frac{N-1}{N^2+1}(\alpha_x - \triangle C)\right) \leq 0 \iff \alpha_x + N\triangle C < 0,$$

which is a contradiction since $\alpha_x > 0$, $\triangle C \geq 0$, and $N \geq 2$. Therefore, we assume that $x > 0$. Solving (EC.46a) and (EC.46c) gives

$$\begin{aligned} f &= \frac{N-1}{N^2+MN+M+1}\left(\alpha_x - (MN+M+1)\triangle C\right), \\ x &= \frac{N+1}{N^2+MN+M+1}\left(\alpha_x + N^2\triangle C\right). \end{aligned}$$

Substituting for x yields

$$\text{(EC.46b)} \iff (MN+M+1)\triangle C \leq \alpha_x \leq (MN+M+1)\triangle C + \frac{(N^2+MN+M+1)(N-1)}{N^2-2\sqrt{N}+1}k.$$

Substituting for f yields

$$\begin{aligned} \text{(EC.46d)} &\iff (MN+M+1)\triangle C < \alpha_x \text{ and } \alpha_x \leq \begin{cases} \frac{N(M+1)}{M+N}\triangle C + \left(N + \frac{M+1}{M+N}\right)k, & \text{if } \alpha_x \leq Nk, \\ \zeta_2, & \text{if } \alpha_x > Nk, \end{cases} \\ &\iff (MN+M+1)\triangle C < \alpha_x \text{ and } \alpha_x \leq \begin{cases} Nk, & \text{if } \alpha_x \leq Nk, \\ \zeta_2, & \text{if } \alpha_x > Nk, \end{cases} \\ &\iff (MN+M+1)\triangle C < \alpha_x \leq \zeta_2. \end{aligned}$$

The first equivalence is due to the fact that $k-(\alpha_x - Nk) \geq \eta_3 \iff \alpha_x \leq Nk$. The second equivalence is due to the fact that $\triangle C \geq 0$ and $k > 0$. Next, using the fact that $N \geq 2, \triangle C \geq 0, k > 0$, we obtain

$$\zeta_2 < (MN+M+1)\triangle C + \frac{(N^2+MN+M+1)(N-1)}{N^2 - 2\sqrt{N}+1}k,$$

from which it follows that (f,x) satisfies (EC.46) if and only if $(f,x) \in Q_3$ and $(MN+M+1)\triangle C \leq \alpha_x \leq \zeta_2$.

Case (iii): $y_j(f\mathbf{1}, x\mathbf{1}) = k$. From Propositions EC.3 and EC.4, we infer that (f,x) is a symmetric equilibrium if and only if

$$f \geq -(\alpha_x - \triangle C - Mx) + (N+1)k, \tag{EC.47a}$$

$$(N+1)k \leq \alpha_x - \triangle C - Mx, \tag{EC.47b}$$

$$x = \frac{1}{M+1}\left[\alpha_x - Nk\right]_0^\infty, \tag{EC.47c}$$

$$f - \triangle C \geq \begin{cases} k-(\alpha_x - Nk), & \text{if } \alpha_x < Nk, \\ \eta_2, & \text{if } \alpha_x \geq \left(1+\frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right)Nk, \\ \eta_1, & \text{otherwise.} \end{cases} \tag{EC.47d}$$

We divide the analyses into three cases depending on the value of α_x.

Suppose $0 < \alpha_x \leq Nk$. Then, (EC.47c) implies $x = 0$. However, substituting into (EC.47b) implies that $\alpha_x - Nk \geq k + \triangle C > 0$ which is a contradiction. Therefore, there does not exist an equilibrium such that $0 < \alpha_x \leq Nk$.

Suppose $Nk < \alpha_x \leq \left(1+\frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right)Nk$. Then, (EC.47c) implies $x = \frac{1}{M+1}(\alpha_x - Nk)$. Substituting for x yields

$$\text{(EC.47a)} \iff f \geq \triangle C - \frac{1}{M+1}(\alpha_x - Nk) + k,$$

$$\text{(EC.47b)} \iff \triangle C - \frac{1}{M+1}(\alpha_x - Nk) + k \leq 0.$$

From (EC.47d), we infer that $f \geq \triangle C + \eta_1$. Since $N \geq 2 \implies \eta_1 \geq k - \frac{\alpha_x - Nk}{M+1}$, it follows that the symmetric equilibria are characterized by

$$x = \frac{1}{M+1}(\alpha_x - Nk) \text{ and } f \geq \triangle C + \eta_1 \text{ and } \triangle C - \frac{1}{M+1}(\alpha_x - Nk) + k \leq 0. \tag{EC.48}$$

Suppose $\left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk < \alpha_x$. Then, we again have

$$\begin{aligned}
\text{(EC.47a)} &\iff f \geq \triangle C - \frac{1}{M+1}(\alpha_x - Nk) + k, \\
\text{(EC.47b)} &\iff \triangle C - \frac{1}{M+1}(\alpha_x - Nk) + k \leq 0.
\end{aligned}$$

From (EC.47d), we infer that $f \geq \triangle C + \eta_2$. Since $N \geq 2 \implies \eta_2 \geq k - \frac{\alpha_x - Nk}{M+1}$, it follows that the symmetric equilibria are characterized by

$$x = \frac{1}{M+1}(\alpha_x - Nk) \text{ and } f \geq \triangle C + \eta_2 \text{ and } \triangle C - \frac{1}{M+1}(\alpha_x - Nk) + k \leq 0. \tag{EC.49}$$

By combining the characterizations in (EC.48) and (EC.49), we infer that (f, x) satisfies (EC.47) if and only if $(f, x) \in Q_4$ and $(M+1)(\triangle C + k) + Nk \leq \alpha_x$.

EC.1.5. Stackelberg Equilibrium

THEOREM EC.2. *Suppose followers' forward positions $\mathbf{f} = \mathbf{0}$. Let $X \subseteq \mathbb{R}_+$ denote the set of symmetric leader reactions, i.e., for each $x \in X$ and $i \in M$,*

$$\psi_i(x; x\mathbf{1}, \mathbf{0}) \geq \psi_i(\bar{x}; x\mathbf{1}, \mathbf{0}), \quad \forall \bar{x} \in \mathbb{R}_+.$$

Let:

$$\begin{aligned}
x_1 &= \frac{1}{M+1}[\alpha_x]_0^\infty, \\
x_2 &= \frac{1}{M}(\alpha_x - \triangle C), \\
x_3 &= \frac{1}{M+1}[\alpha_x + N\triangle C]_0^\infty, \\
x_4 &= \frac{1}{M+1}[\alpha_x - Nk]_0^\infty.
\end{aligned}$$

Then,

$$X = \left\{ x \in \mathbb{R}_+ \;\middle|\; \begin{array}{l} x = x_1 \text{ if } \alpha_x < (M+1)\triangle C, \\ \text{or } x = x_2 \text{ if } (M+1)\triangle C \le \alpha_x \le \min\left((MN+M+1)\triangle C, \max\left(\zeta_1, \triangle C + M(N+1)k\right)\right), \\ \text{or } x = x_3 \text{ if } (MN+M+1)\triangle C < \alpha_x \le \zeta_2, \\ \text{or } x = x_4 \text{ if } \alpha_x \ge \begin{cases} Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k), & \text{if } (\sqrt{N+1}-1)^2 \triangle C < Nk, \\ Nk + (M+1)\left(\triangle C + \sqrt{Nk\triangle C}\right), & \text{otherwise.} \end{cases} \end{array} \right\}.$$

where

$$\zeta_1 := MNk + (M+1)\triangle C + 2M\sqrt{Nk\triangle C},$$
$$\zeta_2 := (MN+M+1)\triangle C + \frac{(M+1)\sqrt{N+1}}{2(\sqrt{N+1}-1)}\left(Nk - \left(\sqrt{N+1}-1\right)^2 \triangle C\right).$$

Moreover, for each $x \in X$,

$$y_j(\mathbf{0}, x\mathbf{1}) = 0 \iff x = x_1 \text{ or } x_2,$$
$$0 < y_j(\mathbf{0}, x\mathbf{1}) < k \iff x = x_3,$$
$$y_j(\mathbf{0}, x\mathbf{1}) = k \iff x = x_4.$$

Proof. The result is obtained by substituting $f = 0$ into Proposition EC.4 and simplifying the inequalities in X. For the case of $x = x_1$, we have

$$f - \triangle C < -\frac{\alpha_x}{M+1} \iff \alpha_x < (M+1)\triangle C.$$

For the case of $x = x_2$, we have

$$\begin{aligned} & -\frac{\alpha_x}{M+1} \le f - \triangle C \le \min\left(-\frac{\alpha_x}{MN+M+1}, \max\left(\eta_4, -\alpha_x + M(N+1)k\right)\right) \\ \iff & -\frac{\alpha_x}{M+1} \le -\triangle C \text{ and } -\triangle C \le -\frac{\alpha_x}{MN+M+1} \text{ and } -\triangle C \le \max\left(\eta_4, -\alpha_x + M(N+1)k\right) \\ \iff & (M+1)\triangle C \le \alpha_x \text{ and } \alpha_x \le (MN+M+1)\triangle C \text{ and } \alpha_x \le \max\left(\zeta_1, \triangle C + M(N+1)k\right) \\ \iff & (M+1)\triangle C \le \alpha_x \le \min\left((MN+M+1)\triangle C, \max\left(\zeta_1, \triangle C + M(N+1)k\right)\right), \end{aligned}$$

where the second equivalence is due to the fact that $-\triangle C \leq \eta_4 \iff \alpha_x \leq \zeta_1$. For the case of $x = x_3$, we have

$$
\begin{aligned}
& -\frac{\alpha_x}{MN+M+1} < f - \triangle C \leq \max\left(\eta_3, k - (\alpha_x - Nk)\right) \\
\iff & -\frac{\alpha_x}{MN+M+1} < -\triangle C \leq \begin{cases} k - (\alpha_x - Nk), & \text{if } \alpha_x \leq Nk, \\ \eta_3, & \text{if } \alpha_x > Nk, \end{cases} \\
\iff & (MN+M+1)\triangle C < \alpha_x \leq \begin{cases} Nk, & \text{if } \alpha_x \leq Nk, \\ \zeta_2, & \text{if } \alpha_x > Nk, \end{cases} \\
\iff & (MN+M+1)\triangle C < \alpha_x \leq \zeta_2.
\end{aligned}
$$

The first equivalence is due to the fact that $k - (\alpha_x - Nk) \geq \eta_3 \iff \alpha_x \leq Nk$. The second equivalence is due to the fact that $\triangle C \geq 0$ and $k > 0$. For the case of $x = x_4$, we have

$$
f - \triangle C \geq \begin{cases} k - (\alpha_x - Nk), & \text{if } \alpha_x < Nk, \\ \eta_2, & \text{if } \alpha_x \geq \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \\ \eta_1, & \text{otherwise.} \end{cases}
$$

Suppose $\alpha_x < Nk$. Then, the above inequality implies that $-\triangle C \geq k - (\alpha_x - Nk) \implies \alpha_x \geq \triangle C + (N+1)k > Nk$, which is a contradiction. Henceforth, we assume that $\alpha_x \geq Nk$, and obtain

$$
\begin{aligned}
& -\triangle C \geq \begin{cases} \eta_2, & \text{if } \alpha_x \geq \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \\ \eta_1, & \text{if } Nk \leq \alpha_x < \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \end{cases} \\
\iff & \alpha_x \geq \begin{cases} Nk + (M+1)\left(\triangle C + \sqrt{Nk\triangle C}\right), & \text{if } \alpha_x \geq \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \\ Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k), & \text{if } Nk \leq \alpha_x < \left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right) Nk, \end{cases} \\
\iff & \alpha_x \geq \begin{cases} Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k), & \text{if } (\sqrt{N+1}-1)^2 \triangle C < Nk, \\ Nk + (M+1)\left(\triangle C + \sqrt{Nk\triangle C}\right), & \text{otherwise.} \end{cases}
\end{aligned}
$$

The last equivalence is due to the fact that $\frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C+k) < \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}Nk \iff (\sqrt{N+1}-1)^2\triangle C < Nk$.

EC.2. Proofs of Structural Results

EC.2.1. Proof of Lemma 1

From Proposition EC.3, note that $0 < y_j(f\mathbf{1}, x\mathbf{1}) < k$ if and only if $f = \frac{N-1}{N^2+1}(\alpha_x - \triangle C - Mx)$. Substituting into the follower productions from Proposition EC.2 gives

$$y_j(f\mathbf{1}, x\mathbf{1}) = \frac{N}{N^2+1}\xi,$$

which is strictly increasing in ξ. Since $\xi \le \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k$, we obtain

$$\begin{aligned}
\bar{y} &= \left(1 - \frac{N-2\sqrt{N}+1}{N^2-2\sqrt{N}+1}\right)k \\
&\ge \left(1 - \frac{N+1}{N^2-2\sqrt{N}}\right)k \\
&\ge \left(1 - \frac{1}{N}\frac{N+1}{N}\frac{N^2}{N^2-2\sqrt{N}}\right)k,
\end{aligned}$$

which gives the first claim.

Next, from Proposition EC.3, $\underline{\xi}$ and $\bar{\xi}$ are given by

$$\begin{aligned}
\underline{\xi} &= (N+1)k, \\
\bar{\xi} &= \frac{(N^2+1)(N-1)}{N^2-2\sqrt{N}+1}k,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\frac{\underline{\xi}-\bar{\xi}}{\underline{\xi}} &= \frac{2(N^2-N\sqrt{N}-\sqrt{N}+1)}{(N^2-2\sqrt{N}+1)(N+1)} \\
&\le \frac{2(N^2+1)}{N(N^2-2\sqrt{N})} \\
&= \frac{2}{N}\frac{N^2+1}{N^2}\frac{N^2}{N^2-2\sqrt{N}},
\end{aligned}$$

which gives the rest of the second claim.

EC.2.2. Proof of Lemma 2

From Proposition EC.4, $\underline{f}$ and $\bar{f}$ are given by

$$\begin{aligned}
\underline{f} &= \triangle C - \frac{\alpha_x}{M+1}, \\
\bar{f} &= \triangle C + \min\left(-\frac{\alpha_x}{MN+M+1}, \max\left(\eta_4, -\alpha_x + M(N+1)k\right)\right).
\end{aligned}$$

The first claim follows from Proposition EC.4. Next,

$$\begin{aligned}
\bar{f} - \underline{f} &= \frac{\alpha_x}{M+1} + \min\left(-\frac{\alpha_x}{MN+M+1}, \max\left(\eta_4, -\alpha_x + M(N+1)k\right)\right) \\
&\leq \frac{\alpha_x}{M+1} - \frac{\alpha_x}{MN+M+1} \\
&= \frac{MN\alpha_x}{(M+1)(MN+M+1)} \\
&\leq \frac{MN\alpha_x}{M^2(N+1)} \\
&= \frac{\alpha_x}{M}\frac{N}{N+1},
\end{aligned}$$

which gives the second claim.

EC.2.3. Proof of Lemma 3

From Proposition EC.4, note that $0 < y_j(f\mathbf{1}, x\mathbf{1}) < k \implies x = x_3$. Substituting into the follower productions from Proposition EC.2 gives

$$y_j(f\mathbf{1}, x\mathbf{1}) = \left[\frac{1}{N+1}\left(\alpha_x + (f - \triangle C) - \frac{M}{M+1}(\alpha_x + N(\triangle C - f))\right)\right]_0^k,$$

which is strictly increasing in f. Note that $x = x_3$ is a reaction if and only if

$$-\frac{\alpha_x}{MN+M+1} < f - \triangle C \leq \max\left(\eta_3, k - (\alpha_x - Nk)\right) \iff -\frac{\alpha_x}{MN+M+1} < f - \triangle C \leq \eta_3,$$

where we used the fact that $\alpha_x > Nk \implies \eta_3 \geq k - (\alpha_x - Nk)$. Since

$$\alpha_x \leq Nk\left(1 + \frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2} + \frac{(M-1)\sqrt{N+1}}{\sqrt{N+1}-1}\right) \implies -\frac{\alpha_x}{NM+M+1} \leq \eta_3,$$

we infer the case for $\bar{y}=0$. Otherwise, substituting for η_3 gives

$$\begin{aligned}\bar{y} &= y_j\left((\triangle C+\eta_3)\mathbf{1}, x_3\mathbf{1}\right)\\ &= k+\frac{\alpha_x-Nk}{N+1}\frac{1}{M+1}\left(\frac{2(M+1)(N+1)-(M+1)(N+2)\sqrt{N+1}}{N(2+(M-1)\sqrt{N+1}}\right)\\ &\geq k-\frac{\alpha_x-Nk}{N}\left(\frac{N+2}{(N+1)(M-1)}\right),\end{aligned}$$

from which we obtain the first claim. From Proposition EC.4, we infer that $\bar{f}=\eta_3$ and $\underline{f}=\eta_1$ when $\alpha_x \leq Nk\left(1+\frac{(M+1)\sqrt{N+1}}{(\sqrt{N+1}-1)^2}\right)$. Therefore, we obtain

$$\begin{aligned}\bar{f}-\underline{f} &= \frac{\alpha_x-Nk}{N}\left(2(\sqrt{N+1}-1\right)\left(\frac{2+(M-1)\sqrt{N+1}-(M+1)}{(M+1)(2+(M-1)\sqrt{N+1}}\right)\\ &\leq 2\left(\frac{\alpha_x-Nk}{N}\right)\left(\frac{\sqrt{N+1}-1}{M-1}\right)\\ &\leq \frac{\alpha_x-Nk}{M\sqrt{N}}2\left(\frac{\sqrt{N+1}}{\sqrt{N}}\frac{M}{M-1}\right),\end{aligned}$$

which gives the second claim.

EC.2.4. Proof of Lemma 4

From Theorem EC.1, note that $0<y_j(f\mathbf{1},x\mathbf{1})<k \implies (f,x)\in Q_3$. Substituting into the follower productions from Proposition EC.2 gives

$$y_j(f\mathbf{1},x\mathbf{1})=\frac{N}{N^2+NM+M+1}\alpha_x,$$

which is strictly increasing in α_x. Since $\alpha_x \leq \zeta_2$, it follows that

$$\begin{aligned}\bar{y} &= \frac{N}{N^2+NM+M+1}\zeta_2\\ &= \left(1-\frac{N+2-2\sqrt{N+1}}{N^2+N+2-2\sqrt{N+1}}\right)k\\ &\geq \left(1-\frac{N}{N^2+N}\right)k\\ &= \left(1-\frac{1}{N}\frac{N}{N+1}\right)k,\end{aligned}$$

from which we obtain the first claim.

Next, from Theorem EC.1, we infer that $\bar{\alpha}_x = \zeta_2$ and $\underset{\sim}{\alpha} = (M+N+1)k$. It is easy to show that $\zeta_2 < (M+N+1)k \iff M < N\sqrt{N+1} - 1$. Moreover,

$$(M+N+1)k - \zeta_2 = \frac{2}{N^2 + (\sqrt{N+1}-1)^2}\left((N^2+N+M+1) - (M+N+1)\sqrt{N+1}\right)k.$$

Therefore, if $\underset{\sim}{\alpha}_x \le \bar{\alpha}_x$, then

$$\begin{aligned}
\frac{\underset{\sim}{\alpha}_x - \bar{\alpha}_x}{\underset{\sim}{\alpha}_x} &= \frac{2}{N^2 + (\sqrt{N+1}-1)^2}\left(1 + \frac{N^2}{M+N+1} - \sqrt{N+1}\right) \\
&\le \frac{2N}{N^2 + (\sqrt{N+1}-1)^2} \\
&\le \frac{2N}{N^2},
\end{aligned}$$

from which we obtain the first part of the second claim. If $\underset{\sim}{\alpha}_x \ge \bar{\alpha}_x$, then

$$\begin{aligned}
\frac{\bar{\alpha}_x - \underset{\sim}{\alpha}_x}{\underset{\sim}{\alpha}_x} &= \frac{2}{N^2 + (\sqrt{N+1}-1)^2}\left(\sqrt{N+1} - 1 - \frac{N^2}{M+N+1}\right) \\
&\le \frac{2}{N^2 + (\sqrt{N+1}-1)^2}\left(\sqrt{N+1}\right) \\
&\le \frac{2}{N^2}\sqrt{N+2} \\
&\le \frac{2}{N\sqrt{N}}\sqrt{\frac{N+2}{N}},
\end{aligned}$$

from which we obtain the rest of the second claim.

EC.2.5. Proof of Lemma 6

From Theorem EC.1, note that $0 < y_j(f\mathbf{1}, x\mathbf{1}) < k \iff (f,x) \in Q_3$. Substituting into the follower productions from Proposition EC.2 gives

$$y_j(f\mathbf{1}, x\mathbf{1}) = \frac{N}{N^2 + MN + M + 1}\left(\alpha_x - (MN+M+1)\triangle C\right),$$

which is strictly increasing in α_x. Since $\alpha_x \le \zeta_2$, it follows that

$$\begin{aligned}
\bar{y} &= \frac{N}{N^2 + MN + M + 1}\left(\zeta_2 - (MN+M+1)\triangle C\right) \\
&= \frac{N^2}{N^2 + (\sqrt{N+1}-1)^2}\left(k - \frac{(\sqrt{N+1}-1)^2}{N}\triangle C\right) \\
&\ge \left(1 - \frac{N+2-2\sqrt{N+1}}{N^2}\right)\left(k - \frac{(\sqrt{N+1}-1)^2}{N}\triangle C\right) \\
&\ge \left(1 - \frac{1}{N}\right)\left(k - \frac{(\sqrt{N+1}-1)^2}{N}\triangle C\right),
\end{aligned}$$

from which we obtain the first claim.

Next, from Theorem EC.1, we infer that if $(\sqrt{N+1}-1)^2\triangle C < Nk$, then $\bar{\alpha}_x = \zeta_2$ and $\underline{\alpha}_x = (M+N+1)k$, and it is straightforward to show that $\zeta_2 < (M+N+1)k$ if and only if the first case in (3) holds. Otherwise, then $\bar{\alpha}_x = \zeta_1$ and $\underline{\alpha}_x = (M+N+1)k$, and it is straightforward to show that $\zeta_1 < (M+N+1)k$ if and only if the second case in (3) holds.

EC.2.6. Proof of Lemma 7

From Theorem EC.2, note that $0 < y_j(\mathbf{0}, x\mathbf{1}) < k \iff x = x_3$. Substituting into the follower productions from Proposition EC.2 gives

$$y_j(\mathbf{0}, x\mathbf{1}) = \frac{1}{(N+1)(M+1)}\left(\alpha_x - (MN+M+1)\triangle C\right),$$

which is strictly increasing in α_x. Since $\alpha_x \le \zeta_2$, it follows that

$$\begin{aligned}\bar{y} &= \frac{1}{(N+1)(M+1)}\left(\zeta_2 - (MN+M+1)\triangle C\right)\\ &= \left(1+\frac{1}{\sqrt{N+1}}\right)\frac{k}{2},\end{aligned}$$

from which we obtain the first claim.

Next, from Theorem EC.2, we infer that $\bar{\alpha}_x = \zeta_2$ and $\underline{\alpha}_x = Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k)$. It is straightforward to show that $\bar{\alpha}_x \ge \underline{\alpha}_x$.

EC.2.7. Proof of Lemma 9

From Theorem EC.2, note that $0 < y_j(\mathbf{0}, x\mathbf{1}) < k \iff x = x_3$. Substituting into the follower productions from Proposition EC.2 gives

$$y_j(\mathbf{0}, x\mathbf{1}) = \frac{1}{(N+1)(M+1)}\left(\alpha_x - (MN+M+1)\triangle C\right),$$

which is strictly increasing in α_x. Since $\alpha_x \le \zeta_2$, it follows that

$$\begin{aligned}\bar{y} &= \frac{1}{(N+1)(M+1)}\left(\zeta_2 - (MN+M+1)\triangle C\right)\\ &= \left(1+\frac{1}{\sqrt{N+1}}\right)\frac{1}{2}\left(k - \frac{(\sqrt{N+1}-1)^2}{N}\triangle C\right),\end{aligned}$$

from which we obtain the first claim.

Next, from Theorem EC.2, we infer that, if $(\sqrt{N+1}-1)^2\triangle C < Nk$, then $\bar{\alpha}_x = \zeta_2$ and $\underline{\alpha}_x = Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}(\triangle C + k)$, and it is straightforward to show that $\bar{\alpha}_x \geq \underline{\alpha}_x$. Otherwise, then $\bar{\alpha}_x = \zeta_1$ and $\underline{\alpha}_x = Nk + (M+1)(\triangle C + \sqrt{Nk\triangle C})$, and it is straightforward to show that $\bar{\alpha}_x \geq \underline{\alpha}_x$.

EC.2.8. Proof of Lemma 10

The proof proceeds in three steps. In step 1, we compute an equilibria with the smallest (resp. largest) market production in the forward (resp. Stackelberg) market. In step 2, we compute an equilibria with the smallest (resp. largest) social welfare in the forward (resp. Stackelberg) market. In step 3, we show that the worst case ratios of productions and efficiencies are both strictly increasing in α_x. The bounds in the lemma are obtained by evaluating those ratios at $\alpha_x = \bar{\alpha}_x$.

Step 1: We compute an equilibria with the smallest (resp. largest) market production in the forward (resp. Stackelberg) market. First, we tackle the forward market. Substituting $\triangle C = 0$ into Theorem EC.1, we infer that $(f,x) \in Q$ if and only if $(f,x) \in Q_3$ or $(f,x) \in Q_4$. By substituting into Theorem EC.2, and using the fact that $y_j(f\mathbf{1}, x\mathbf{1}) = 0$ for all $(f,x) \in Q_4$, we obtain the following market productions:

$$Mx + Ny_j(f\mathbf{1}, x\mathbf{1}) = \begin{cases} \frac{1}{N^2+MN+M+1}\left(N^2+MN+M\right)\alpha_x, & \text{if } (f,x) \in Q_3, \\ \frac{1}{M+1}\left(M\alpha_x + Nk\right), & \text{if } (f,x) \in Q_4. \end{cases}$$

Note that

$$\begin{aligned}
&\frac{1}{M+1}\left[M\alpha_x + Nk\right] \\
&= \frac{1}{N^2+MN+M+1}\left[(N^2+MN+M)\alpha_x + \frac{-N^2-MN}{M+1}\alpha_x + \frac{N(N^2+MN+M+1)}{M+1}k\right] \\
&\leq \frac{1}{N^2+MN+M+1}\left[(N^2+MN+M)\alpha_x + \frac{N(M+1)(1-N-M)}{M+1}k\right] \\
&\leq \frac{1}{N^2+MN+M+1}\left(N^2+MN+M\right)\alpha_x,
\end{aligned}$$

where the first inequality is due to the fact that $\alpha_x \geq \underline{\alpha}_x$ and the second inequality is due to the fact that $M \geq 1$, $N \geq 2$, and $k > 0$. Therefore, we infer that the smallest equilibrium production in the forward market is given by

$$\begin{aligned} y_F &= k, \\ x_F &= \frac{1}{M+1}(\alpha_x - Nk). \end{aligned}$$

Next, we tackle the Stackelberg market. Substituting $\triangle C = 0$ into Theorem EC.2, we infer that $(0, x_s) \in X(0)$ if and only if $x_s = x_3$ or $x_s = x_4$. Suppse

$$\alpha_x < Nk + \frac{N(M+1)}{2(\sqrt{N+1}-1)}k. \tag{EC.50}$$

Then, from Theorem EC.2, we conclude that $x_s = x_3$ is the only Stackelberg equilibrium, and hence it is also the equilibrium with the largest market production. Suppose, instead, that (EC.50) does not hold. By substituting into Proposition EC.2, and using the fact that $y_j(\mathbf{0}, x_4\mathbf{1}) = 0$, we obtain the following market productions:

$$Mx_s + Ny_j(\mathbf{0}, x_s\mathbf{1}) = \begin{cases} \frac{MN+M+N}{(M+1)(N+1)}\alpha_x, & \text{if } x_s = x_3, \\ \frac{1}{M+1}\left(M\alpha_x + Nk\right), & \text{if } x_s = x_4. \end{cases}$$

Note that

$$\begin{aligned} &\frac{MN+M+N}{(M+1)(N+1)}\alpha_x \\ &= \frac{1}{M+1}\left[M\alpha_x + \frac{N}{N+1}\alpha_x\right] \\ &\geq \frac{1}{M+1}\left[M\alpha_x + \frac{N}{N+1}\left(N + \frac{N(M+1)}{2(\sqrt{N+1}-1)}\right)k\right] \\ &\geq \frac{1}{M+1}\left[M\alpha_x + \frac{N}{N+1}\left(N+1\right)k\right] \\ &= \frac{1}{N+1}\left[M\alpha_x + Nk\right], \end{aligned}$$

where the first inequality is due to the fact that (EC.50) does not hold and the second inequality is due to the fact that $M \geq 1$ and $N \geq 2$. Therefore, we infer that the largest equilibrium production in the Stackelberg market is given by

$$y_S = \frac{1}{N+1}\left(\alpha_x - Mx_s\right),$$

$$x_S = \frac{1}{M+1}\alpha_x.$$

Step 2: We compute the equilibria with the smallest (resp. largest) social welfare in the forward (resp. Stackelberg) market. Substituting the demand function into the social welfare gives

$$\begin{aligned}\mathsf{SW}(y,x) &= \beta\left(\alpha_x\left(Mx+Ny\right) - \triangle CNy - \frac{1}{2}\left(Mx+Ny\right)^2\right)\\ &= \beta\left(\alpha_x\left(Mx+Ny\right) - \frac{1}{2}\left(Mx+Ny\right)^2\right),\end{aligned}$$

where the second equality is obtained by substituting $\triangle C = 0$. Given any two equilibrium productions (y,x) and (y',x'), we have

$$\begin{aligned}&\mathsf{SW}(y,x) \geq \mathsf{SW}(y',x')\\ \Longleftrightarrow\ &\alpha_x(Mx+Ny) - \frac{1}{2}(Mx+Ny)^2 \geq \alpha_x(Mx'+Ny') - \frac{1}{2}(Mx'+Ny')^2\\ \Longleftrightarrow\ &\frac{1}{2}\left((Mx+Ny)-(Mx'+Ny')\right)\left(\alpha_x - (Mx+Ny) + \alpha_x - (Mx'+Ny')\right) \geq 0\\ \Longleftrightarrow\ &\frac{1}{2}\left((Mx+Ny)-(Mx'+Ny')\right)\left(\frac{1}{\beta}\left(P(Mx+Ny)-C\right) + \frac{1}{\beta}\left(P(Mx'+Ny')-C\right)\right) \geq 0\\ \Longleftrightarrow\ &Mx+Ny \geq Mx'+Ny',\end{aligned}$$

where the last equivalence follows from the fact that, since (y,x) and (y',x') are equilibrium productions, the profit margins $P(Mx+Ny)-C>0$ and $P(Mx'+Ny')-C>0$. Therefore, the equilibria with the smallest (resp. largest) social welfare in the forward (resp. Stackelberg) market are those with the smallest (resp. largest) market productions, which were obtained in step 1.

Step 3: We show that the worst-case ratios of productions and social welfares are strictly increasing in α_x. From step 1, the ratio of productions is bounded from above by

$$r_P := \frac{Mx_S+Ny_S}{Mx_F+Ny_F}.$$

Taking derivatives gives

$$\begin{aligned}\frac{\partial r_P}{\partial \alpha_x} &= \frac{(Mx_F+Ny_F)\left(M\frac{\partial x_S}{\partial \alpha_x} + N\frac{\partial y_S}{\partial \alpha_x}\right) - (Mx_S+Ny_S)\left(M\frac{\partial x_F}{\partial \alpha_x} + N\frac{\partial y_F}{\partial \alpha_x}\right)}{(Mx_F+Ny_F)^2}\\ &= \frac{\frac{N(MN+M+N)k}{(M+1)^2(N+1)}}{(Mx_F+Ny_F)^2}\\ &> 0.\end{aligned}$$

Next, the ratio of social welfares is bounded from above by

$$r_W := \frac{\mathsf{SW}(y_S, x_S)}{\mathsf{SW}(y_F, x_F)}.$$

Taking derivatives gives

$$\begin{aligned}
\frac{\partial r_W}{\partial \alpha_x} &= \frac{\mathsf{SW}(y_F, x_F)\frac{\partial \mathsf{SW}(y_S, x_S)}{\partial \alpha_x} - \mathsf{SW}(y_S, x_S)\frac{\partial \mathsf{SW}(y_F, x_F)}{\partial \alpha_x}}{\mathsf{SW}(y_F, x_F)^2} \\
&= \frac{\frac{\beta}{2(M+1)^4(N+1)^2}(MN+M+N)(MN+M+N+2)(\alpha_x - Nk)Nk\alpha_x}{\mathsf{SW}(y_F, x_F)^2} \\
&> 0,
\end{aligned}$$

where the inequality is due to $\alpha_x \geq \underline{\alpha}_x > Nk$. Therefore, r_P and r_W are both strictly increasing in α_x over $[\underline{\alpha}_x, \bar{\alpha}_x]$. By substituting $\alpha_x = \bar{\alpha}_x$ into r_P and r_W, we obtain the desired result.